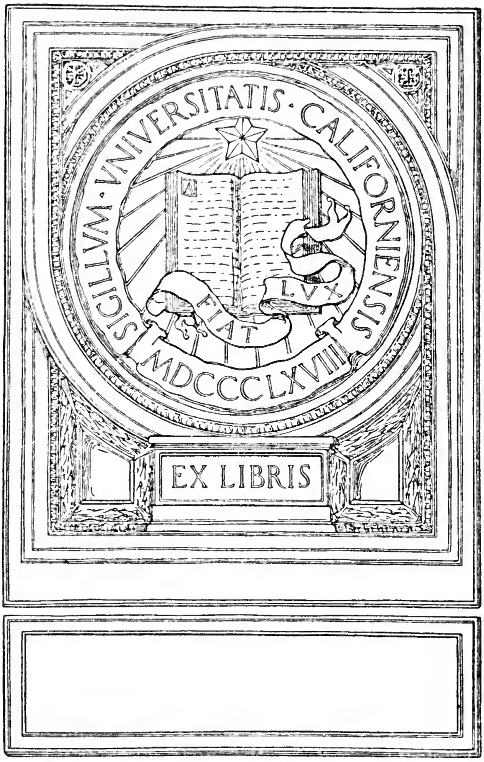
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THE MECHANICAL PRINCIPLES OF THE AEROPLANE

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PREFACE

The Dynamics of the Aeroplane is a subject now being taught at several of our universities and colleges; there can be no doubt that before long it will form part of the curricula of pass and honours schools of mathematics or engineering at most institutions of university rank. The student who approaches the subject from the mathematical point of view must get a clear grasp of the principles underlying the theory of resisted motion, three-dimensional dynamics, and the theory of fluid motion with free stream lines. Although several excellent books on aeroplanes have already been published, notably Bairstow's monumental treatise on *Applied Aerodynamics*—a veritable mine of information on all subjects aeronautical—there should yet be room for the present work, in which an attempt has been made to deal with the subject in such a way as to emphasise the fundamental ideas on which the theory of aeroplane motion is built. It is for this reason that so much space has been devoted to such questions as the theory of dimensions, moving axes, and the hydrodynamics of a perfect fluid in two dimensions. The *a priori* study of aeroplanes in Chapter VII., obviously inspired by Bryan's classical work, should help to make clear the reasons for the forms of aeroplanes now in use.

As the subject is yet in its infancy, and there is great scope for further research, several investigations have been included with a view to indicating lines along which much mathematical work can be done. Many of the exercises have been set for this purpose.

The notation employed in this book is one that has been carefully devised in consultation with Professor Bryan, so that it may be used with comfort in such complicated investigations as arise in connection with the general motion of the aeroplane. Appendices have been added to Chapters III. and IV. indicating the notation used in the *Technical Reports of the Advisory Committee for Aeronautics*, where so much that is fundamental in the behaviour of aerofoils and aeroplanes has been published.

My sincere thanks are due to Professor Bryan for his inspiration, and for his help in many respects. My best thanks are also due to my colleague, Mr. C. W. Gilham, M.A., to Mr. D. Williams, B.Sc., of the National Physical Laboratory, and to Dr. H. Levy, of the Imperial College, South Kensington, for their kind help in reading the proofs. I cannot hope that all errors have been eliminated; but, while assuming sole responsibility for the views expressed, I feel that the numerous hints given me by these friends have helped to make the presentation clearer and more correct. I also wish to thank Messrs. Constable for their kind permission to use the phugoid chart from Lanchester's *Aerodonetics*.

S. BRODETSKY.

THE UNIVERSITY, LEEDS.

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THE MECHANICAL PRINCIPLES OF THE AEROPLANE

INTRODUCTION

POSSIBILITY OF FLIGHT IN A HEAVIER-THAN-AIR MACHINE

1. If a body is held at rest in still air it experiences a certain pressure on every part of its surface exposed to the air. In elementary books on hydrostatics it is shown that the pressure on any small portion of the surface is in a normal direction, its amount per unit area being, however, independent of the direction of this normal and equal to the so-called pressure of the air in the immediate vicinity. To find the resultant pressure on the body we must add up vectorially all the pressures on the large number of small areas into which the surface of the body can be divided.

The problem becomes very simple in ordinary cases, for if the body has some moderate size we can assume the air density to be constant all round it, and then we use the **Principle of Archimedes** which states that *the total pressure exerted by a homogeneous fluid on a body, both being at rest, with the latter totally or partially immersed in the fluid, is an upward vertical force or buoyancy equal to the weight of the fluid displaced.* Thus the buoyancy is obtained by multiplying the volume of the immersed part of the body into the weight of unit volume of the fluid.

The principle of Archimedes is sufficient to enable us to discuss the condition for the equilibrium suspension of a body in still air, as, *e.g.*, in the case of a balloon at rest. We have only to equate the weight of the balloon, including the gas it contains and all its appendages, to the weight of an equal volume of air of a density equal to that of the air in the vicinity of the balloon. Or, if a ship is at rest in still water, the weight of the ship and all it contains must be equal to the weight of a volume of water equal to the volume of the part immersed. (In the case of a heavy body like a ship, the buoyancy of the air can be neglected.)

But the problem is entirely different when we wish to consider the case in which a body is moving through a fluid. We can either have the

body at rest and the fluid in motion or both the body and the fluid can be moving. The latter includes what is often loosely described as the case of "a body moving through a fluid at rest," since, of course, the motion of the body must, and does, cause motion in the fluid.

Mathematically speaking, we can calculate the force exerted by the fluid on the body by adding up vectorially the pressures on the various parts of the exposed surface. But this is generally of little help to us, because in most cases we are quite ignorant of the pressures set up in the fluid. But of one thing we can be certain *a priori*, namely, that *the force exerted by a fluid on an immersed body when there is relative motion is in the nature of a resistance, tending to reduce the relative motion.* This is the inference derived from such ordinary experiences as walking against a wind, or trying to keep a boat still in a current of water.

2. **Aerodynamics : Mathematical.**—It is the problem of **aerodynamics** to investigate the pressure experienced when a body moves through the air, or, more generally, when there is relative motion of the body with reference to the air, which may have a motion of its own. Two main methods are available. We can attempt to solve the problem by means of **mathematical analysis.** As is generally the case in the application of mathematics to a physical problem, we are forced to introduce assumptions with regard to the nature of the fluid, the shape of the body, and the type of relative motion, that may not be quite justifiable, in order to make a solution by mathematical methods at all possible. We shall see that the application of hydrodynamical methods to air pressures in aerodynamics is at present very restricted in scope, and that considerable progress has yet to be made in the development of powerful mathematical instruments of investigation before we shall be in a position to attack the problem adequately.

Experimental.—Recourse must therefore be had to another method, which consists of **experimental research** on the air pressures on bodies of different shapes and sizes under various conditions of relative motion. Experimental aerodynamics has the advantages and disadvantages of all experimental methods. On the one hand, very elaborate apparatus, involving considerable expense, is required, and long series of careful observations are necessary. But, on the other hand, the results obtained can be made applicable to actual conditions of flight by reproducing these conditions as nearly as possible in the experimental research. It is, of course, clear that experimental investigation is possible no matter what shape is given to the body under discussion, whereas the mathematical methods at present yield results only in very simple cases.

3. **Rigid Dynamics.**—If now we have sufficient information aerodynamically to be able to write down the forces due to air pressure acting on a given body possessed of given relative motion in air in a stated condition of density and motion, we can at once write down the mathematical formulation of the *dynamical problem* of the motion of the body. We now have a purely dynamical investigation to carry out: *a body is moving under forces that are known at any instant and in a given position—to find the motion.* In practice we have a problem in rigid dynamics, and it is the function of the **Rigid Dynamics of the Aeroplane** to investigate the motion of the aeroplane, assuming the knowledge supplied by theoretical and experimental aerodynamics.

4. **Uniform Motion: Statical Equilibrium.**—Let us examine

how a body can be made to move through air, with uniform horizontal velocity but with no rotation of any sort. We suppose that any motion that there is in the air itself, *and such motion must exist*, is due solely to the disturbance caused by the moving body under consideration. If then the body moves quite uniformly, and this has been the case for a considerable time, we can safely assume that the forces due to the air pressures are the same from instant to instant.

The problem thus becomes one of statical equilibrium as far as the totality of forces acting on the moving body are concerned. Since there are no changes in the state of motion it follows that the forces acting must balance. These forces are:

(i) the weight of the body, acting vertically downwards at its centre of gravity;

(ii) the resultant air pressure; and

(iii) any other forces that may be applied in order to maintain the uniform motion.

Let us suppose that the moving body possesses a plane of symmetry, and that this plane is vertical during the motion, the direction of motion lying in the plane. The centre of gravity is of course in the plane of symmetry, and by the fact of the existence of the symmetry it follows that the resultant air pressure must be some system of forces in this plane. The applied forces must also be in this plane; and so we have a problem in two-dimensional statics of a rigid body.

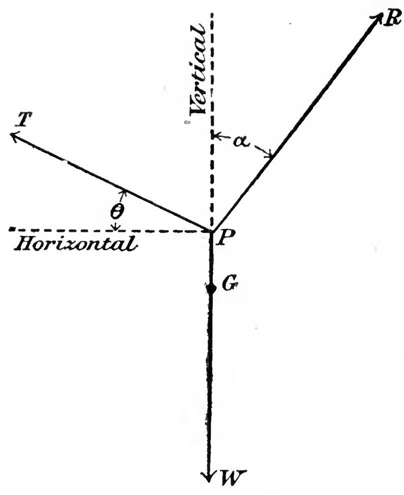

FIG. 1.—Statical Equilibrium: Uniform Motion.

5. The resultant air pressure is some system of forces in the plane of symmetry. We know that a system of forces in one plane can be represented by a single force of the right intensity acting along the correct line of action, except in the special case when the forces are so balanced that they only produce a rotational effect, represented mechanically by a couple. This exceptional case need not be considered here, for we know by experience that resultant air pressures are never merely rotational in effect. Thus we can suppose that the air pressure can be represented by a force acting along some line in the vertical plane of symmetry.

The weight and the resultant air pressure thus meet in a point (or are parallel, but *we do not find that we ever get a vertical air pressure as the result of horizontal motion of the body*). Let Fig. 1 represent the vertical plane of symmetry, G being the centre of gravity. The weight W acts in the vertical line through G. The resultant air pressure, R, meets this line at some point, say P, and makes a certain angle α with the upward vertical. For statical equilibrium the applied forces must balance W and R: hence they can be represented by a force T, which passes through P and also lies in the plane of symmetry. Let T make an angle θ with the

direction of motion. The conditions of equilibrium by Lami's theorem in statics are as follows:

$$\frac{W}{\sin\left(a+\frac{\pi}{2}-\theta\right)}=\frac{R}{\sin\left(\frac{\pi}{2}+\theta\right)}=\frac{T}{\sin(\pi-a)},$$

i.e.

$$\frac{W}{\cos(a-\theta)}=\frac{R}{\cos\theta}=\frac{T}{\sin a};$$

giving

$$T=\frac{W\sin a}{\cos(a-\theta)},\quad R=\frac{W\cos\theta}{\cos(a-\theta)}.$$

6. Now it is clearly an advantage to be able to fly with as small a value of T as possible. In fact, in the development of the aeroplane, it was just the difficulty of getting a prime mover which should give the necessary force T without at the same time unduly increasing the weight W, that retarded progress for a long time. Thus it follows that in practice

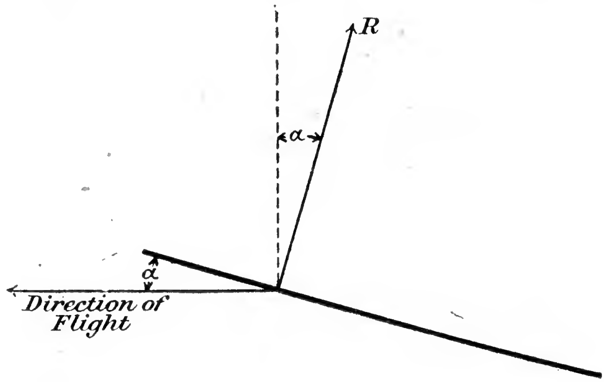

FIG. 2.—Plane: Angle of Attack.

a and θ must be as small as possible. This means that R must be nearly vertical and as large as possible—in fact, nearly equal to W.

Thus for effective horizontal flight it is necessary to arrange the flying body in such a way that the air pressure is nearly vertical, and the vertical part of the air pressure must balance the weight of the body.

7. **Aerofoil: Plane.**—It is, therefore, the task of aerodynamical research to discover such a shape of flying body as will give a nearly vertical air pressure when the direction of motion is horizontal. The form that suggests itself at once is a plane lamina held nearly horizontal and made to move horizontally, so that the air strikes it from beneath (Fig. 2). It is an elementary theorem that in a "perfect fluid" the pressure on an element of surface is always normal—this follows, in fact, from the definition of a perfect fluid. Hence, if the air could be assumed to be "perfect," we could say that a plane lamina inclined at an angle a to the direction of motion would give a resultant air pressure at an angle a with the vertical. The angle a is called the **angle of attack.**

Air is far from behaving like a perfect fluid. Nevertheless, experiment shows that although the resultant air pressure on a lamina moving through

air is not quite normal, it is very nearly so. Thus the aerofoil is suggested—the flying body must consist of a more or less flat lamina or wing, and constructed in such a way that in horizontal motion the lamina makes a small angle with the horizontal direction.

Cambered Aerofoil.—It has been found, however, that the plane lamina is not the best form of wing. It has long been customary to make the wing **cambered,** *i.e.* the section of the wing is curved, and, in fact, the two surfaces of the wing are differently curved, giving the type of section shown in Fig. 3. It is obviously the duty of aerodynamics to investigate the best form of wing section. In this respect mathematical aerodynamics is at present of only limited use—it is the experimental researches that have paved the way for the construction of efficient wings.

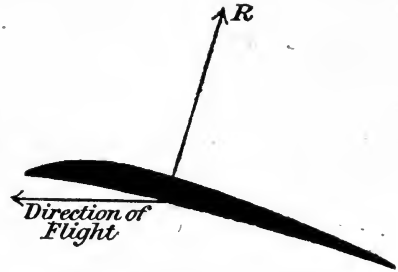

FIG. 3.—Cambered Aerofoil.

8. **Span, Aspect Ratio, Plan Form.**—But not only the form of

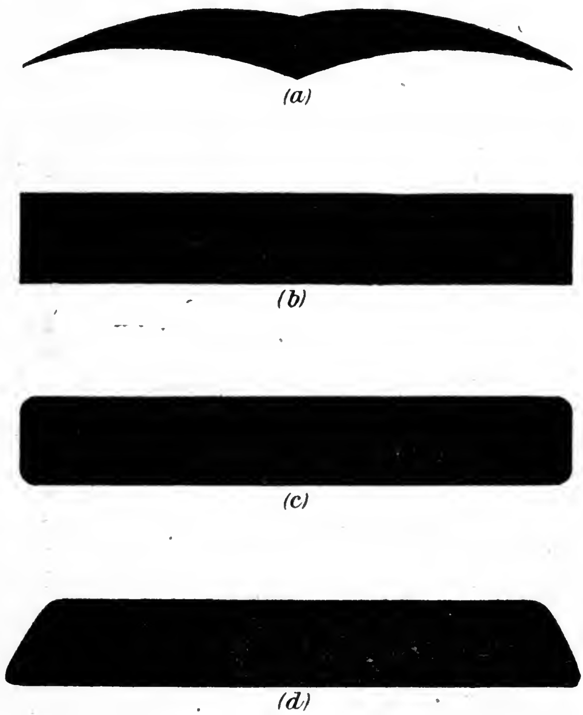

FIG. 4.—Wing Shapes.

wing section is of importance—the general shape of the wing is equally vital. It is necessary to find out the best span for a given width. Here, as in the case of camber, Nature comes to our aid. The forms of the wings

of birds suggest not only the cambered section, but also the great span for a given width. The ratio of span to width is called the **aspect ratio.** Research is required on the most favourable aspect ratio for practical flight. It is found that an aspect ratio of about 6 is best for most purposes.

Information is also desirable concerning the form of the wings in plan. Should it be like the wings of a bird, Fig. 4 (*a*), or rectangular (*b*), or with ends rounded off (*c*), or in the form of a trapezium (*d*), or in some other form that may suggest itself?

9. **Biplane, Stagger.**—A bird's wing is not in one rigid piece. Ignoring the fact that each wing moves independently of the other in the flapping motion by means of which the bird propels itself, we must notice that a bird's wing consists of many parts in the form of feathers, and that the bird makes use of this in its aerial manœuvres. This suggests the use

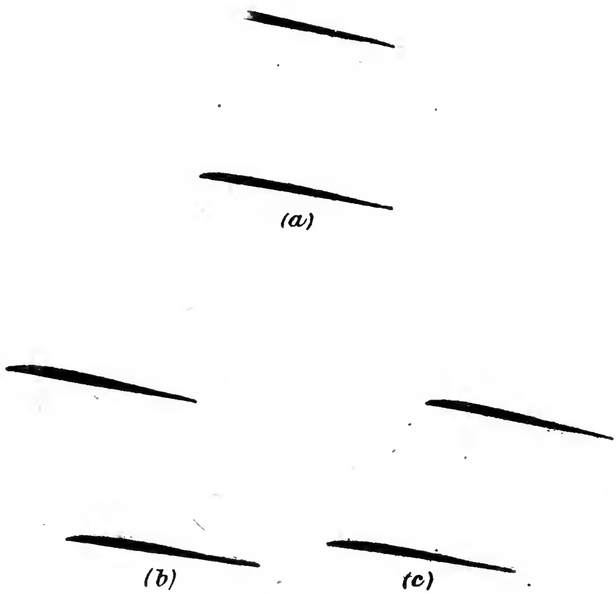

FIG. 5.—Stagger.

of more than one set of wings or parts of wings. We can never hope to rival the marvellous instinct that teaches the bird to use every part of its complicated system of feathers to the best advantage; but we can in some way imitate this action.

Thus experience has shown that the use of two, or even more than two, pairs of wings is a great advantage from certain points of view. In Fig. 5 we have three types of relative configuration of two pairs of wings in section. In 5 (*a*) the wings are just one above the other; but 5 (*b*) shows a formation in which the upper wing is placed in advance of the lower—the arrangement being known as forward **stagger;** whilst 5 (*c*) shows a formation in which the upper wing is behind the lower—this being backward stagger. Which is the best? And, further, what should be the distance between the wings? Finally, should the two pairs of wings be of

exactly the same size, as in some biplanes, or is it an advantage to make the upper wing of wider span, as is the case in other machines?

10. **Body, Fuselage, Landing Gear, Struts.**—We must provide some kind of receptacle for the flyers, and the various controlling and steering devices; we must also provide room for the engine which produces the necessary force or thrust (called T above); and then we must devise some means of getting off the ground and of landing after a flight. Both aerodynamical and engineering considerations thus enter into the problem. Any addition to the wings is a disadvantage from the point of view of flight maintenance, since each addition gives increased air pressure, which will in general be in a backward horizontal direction, thus increasing the force necessary to maintain flight. The engineer and the aerodynamical experimenter must co-operate to devise a form of aeroplane body, engine, landing gear, and system of struts that will produce a strong and yet light structure, with the least possible addition to the necessary propulsive force.

11. **Propeller or Air Screw.**—When this has been done we have to consider further how the energy in the engine is transferred to the

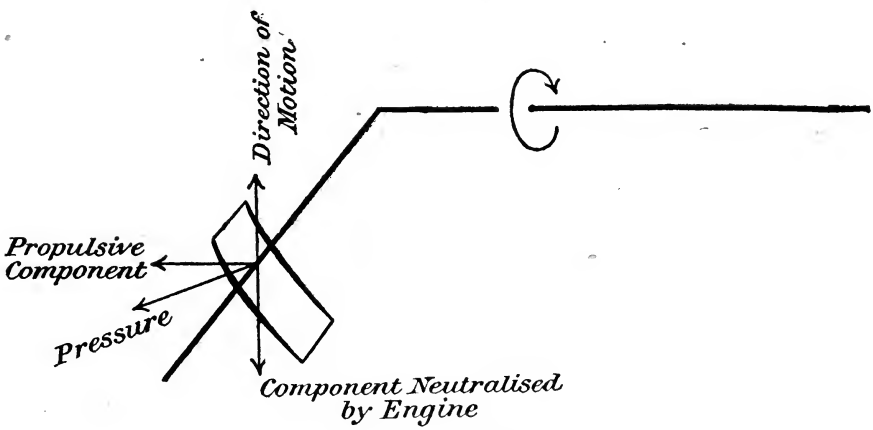

Fig. 6.—Propeller.

aeroplane as a whole for the purposes of producing and maintaining flight. The machine has to be pushed forward from behind, or pulled forward from the front. Just as in walking we need to get a grip on the ground from which to spring off at every step, or just as in a ship the paddles are pushed forward by the water itself by means of a properly devised system of forces from within, so in the air we need to get a grip on the air. This is obtained by means of the propeller or air-screw. The principle is the same as in the propulsion of a ship by means of a screw.

Suppose we have a small plane (or nearly plane) lamina (Fig. 6) mounted on an arm in such a way that the arm rotates about a horizontal axis, whilst at any instant the lamina makes a small angle with the direction in which it is moving at the instant. The effect is to give a nearly horizontal air pressure, which is nearly normal to the lamina. The horizontal component of this pressure has a tendency to move the whole arrangement in a horizontal direction, whereas the other component opposes the rotation of the arm, and has to be neutralised by the torque

in the engine. A propeller consists of a number of such small aerofoils, and aerodynamics must discuss the theory of the propeller, the way to obtain the greatest amount of useful work out of a given amount of work done by the fuel in the engine, and the method of design from the point of view of strength of the blade.

Experimental aerodynamics, aided by the mathematical theory where possible, must thus give us data on the forms and arrangement of wings, shape of body, and type of air-screw to be used in order to obtain the most efficient flight conditions.

12. **Stability.**—For practical purposes this is, however, far from sufficient. In order that an aeroplane shall fly it must be able to rise, it must be able to land, and it must also keep flying safely whilst in the air. This last desideratum did not at first receive the attention it merited. But now all recognise the value of the safety imported by **stability**.

An aeroplane must be stable. Thus if a machine is flying at a uniform rate in a horizontal direction, any disturbance caused by the slight accidents of life, say motion of the flyer, a wind gust, slight variation in the propeller thrust, etc., must not have the effect of destroying utterly the conditions of flight. We must make our machine so that if there is a slight disturbance, the forces called into play will *tend to reproduce the initial state of steady flight.*

But steady flight in a horizontal direction and stability are not sufficient. The machine must be able to climb and descend—in other words, it must be able to fly up or down in directions inclined to the horizontal. Further, it must turn, and controlling devices are required for the purpose. Finally, we must investigate also, as far as possible, the general motion of the aeroplane, not necessarily in a straight line or in a circle. Aeroplanes nowadays do things that are quite difficult to discuss dynamically, and aeroplane mathematics must try to keep in step with these developments.

13. The scientific theory of aeroplanes thus involves several main problems:

(i) aerodynamical investigations by mathematics and by experiment;
(ii) the theory of steady flight in its various possible forms;
(iii) stability;
(iv) the general theory of aeroplane motion.

It is the object of this work to give an account of the mathematical parts of these problems. The theory of engines and the question of strength in the various parts are engineering problems that do not concern us here.

Our method will be as follows. We shall first in Section I. give a brief statement of the general theory of resisted motion, deducing as much as possible from general considerations. We shall then write down the dynamics of a body moving in air, arriving at the theory of steady flight, stability, and including some account of the general problem. In this way we shall indicate what it is that aerodynamical research must investigate. In Section II. we shall give a brief account of the hydrodynamical method of attacking the problem of aerodynamics. Finally, in Section III. we shall apply theoretical and experimental results to the steady flight and stability of aeroplanes, and shall follow this up with an account of the motion of a disturbed aeroplane.

SECTION I

MOTION IN AIR

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14. **The Problem of Resisted Motion.**—In order to decide upon the proper direction for theoretical and experimental investigations in aerodynamics, we must first obtain an idea of what is required by considering the equations of motion of an aeroplane. We at once meet theoretical and practical difficulties. The equations of dynamics are sufficiently complicated even if we deal with only a single rigid body; they would become hopelessly complicated if we had to take into account the small changes that are continually taking place during flight, *e.g.* the bending of the various parts under the strains they are subjected to at the great speeds now in use, the rotation of the propeller relatively to the machine, the change in the quantity of fuel carried, etc. In aviation, therefore, as in other applications of mathematics to practical problems, we must adopt the policy of scientific abstraction. We must ignore the secondary effects introduced by the above and other distracting influences, and treat the aeroplane as if it were a perfectly rigid body. The result is that our equations will not represent the motion with absolute accuracy. On the other hand, we shall have a first, and for most purposes a sufficiently good, approximation to the actual motion. In the light of the information thus obtained it may be possible, if it is thought desirable, to take into account the factors omitted at first.

We have, then, to consider the motion of a rigid body through air. The motion will be affected by the motions that may exist in the air itself. We shall at present, however, make the assumption that the air is of uniform density all round the body, and that it is at rest at points far enough away from the body, the only motion of the air being due to the disturbances caused by the moving body under consideration. Following a common practice, we may call this the *case of still air*.

The air is a compressible, viscous fluid. We have, then, to discover first whether, and how, the compressibility and the viscosity enter into the formulation of the equations governing the motion of a body in air. We can obtain a general idea of the nature and the effect of air-pressure by means of the theory of dimensions.

CHAPTER I

THEORY OF DIMENSIONS

15. **The Numerical Measure of a Physical Quantity.**—One of the first steps in the progress of a branch of physical science consists in the introduction of numerical measures of the physical ideas and conceptions that form the subject-matter of the science. Thus man must have had the notion of motion, the notion of force, the notion of resistance in air, ever since he became conscious of his surroundings. Yet the real science of mechanics was not possible till more exact ideas were obtained in terms of numbers. Let us, then, consider such a phenomenon as a mechanical force, say weight.

We measure the weight of a body by estimating how many times a certain standard weight is contained in it—the method of doing this need not detain us here. Thus we say that a certain body weighs, *e.g.*, 100 lb. In ordinary life we are so accustomed to the standards adopted by convention that we often omit to say what standard we are using. What is the weight of this body? is often answered by the mere number, 100. In giving this number we do not mean to suggest that the weight of the body is the number 100, but that the weight is 100 times a certain well-known and generally accepted standard.

Two facts are to be noted. We use a certain *number*, 100, and a certain standard or *unit*. Now it is obvious that we are not in any way restricted in the choice of a standard or unit of weight. For some purposes we find it convenient to use ounces, say in the purchase of certain articles of food; for other purposes we use tons, as, *e.g.*, in estimating the weight of a bridge. Also in different countries different units of weight are used. There is no unit fixed by Nature—even the metric system does not lay claim to any natural necessity.

But if we choose a different unit of weight, the number representing the weight of a given body must be different. Thus the body which weighs 100 lb. becomes 45·2 kilograms or 0·0446 ton.

Now any *mathematical statement* about the weight of a body can only *refer to the number* representing the weight: mathematics deals only with numbers. It is true that these numbers can refer to various natural phenomena, but an equation involving these phenomena can only be about the numbers and not about the phenomena themselves. If, then, we are to make a mathematics of any physical science, we have really to make statements about the numbers that represent the various ideas and concepts of the science.

16. **Units.**—It follows that unless care is taken to provide for this circumstance there are bound to be confusion and misunderstanding. Thus

suppose one man wishes to discover the law of the fall of a body under gravity. Living in England, he would naturally find the number of feet fallen in various numbers of seconds. By means of careful measurements he therefore finds that if

$$s = \text{distance fallen}$$

and

$$t = \text{time},$$

then

$$s = 16t^2.$$

Now another man, living in France, will do similar experiments and find

$$s = 490t^2.$$

These equations do not agree although they refer to the same phenomenon. The explanation is that each equation is merely numerical. In the first, s is not distance, nor is it distance per second. In each case s represents the *number* of units of distance (*number* of feet or *number* of centimetres). If we take care to state that

$$s = 16t^2,$$

where s is the number of feet fallen, and t is the number of seconds, and

$$s = 490t^2,$$

where s is the number of centimetres fallen, and t is the number of seconds, then there is no contradiction between the two statements.

17. **Equations Independent of the Choice of Units.**—The re-establishing of harmony is, however, not sufficient for scientific purposes. We want more than this. It is important in the mathematics of science to make use of statements that shall be *true under all conditions, no matter what choice of units is made.* Such an equation is readily found in the case of motion under gravity. If we write

$$s = \tfrac{1}{2}gt^2,$$

where s is the number of feet fallen in t seconds, and g is the change each second in the velocity measured in feet per second, then we have a complete scientific statement of perfect generality. For we find that when centimetres are used instead of feet, we again get

$$s = \tfrac{1}{2}gt^2,$$

where s is the number of centimetres fallen in t seconds, and g is the change each second in the velocity measured in centimetres per second. Let us see why this is the case. Suppose that we take the above equation in terms of feet. The equation says that the number of feet in the distance fallen in a time of which t is the number of seconds is obtained by taking one half of the number of feet per second added to the velocity of a falling body each second, and multiplying this into the square of the number of seconds in the time. If we use a different unit of distance, then the number which represents the distance is changed, and so is the number representing the acceleration due to gravity; but both numbers are changed in the same ratio, with the result that the numerical equation given for feet is also true for centimetres.

Let us still consider the formula for distance fallen and suppose that a different unit of time is taken, say minutes. Now the number which represents the time will be changed, and it is clear that the new number is

one-sixtieth of the old number. At the same time, the number representing the acceleration due to gravity is also changed. We can easily argue out, by common sense, the change produced. If a velocity is given as a certain number of feet per second, then it must be sixty times that number in feet per minute. Further, if a certain change of velocity takes place in a second, then sixty times this change takes place in a minute. Thus, when minutes are used to express the acceleration due to gravity we find that the number representing this acceleration in terms of minutes is $(60)^2$ times as great as before. The number $\frac{1}{2}gt^2$ is changed in such a way that g is $(60)^2$ times as great and t is reduced to one 60^{th}: the number is, therefore, unaltered. In the same way we can consider a change of the two units of distance and time, and we find that $s=\frac{1}{2}gt^2$ is again correct.

18. **Absolute Units.**—An equation which is true, no matter what units are taken in order to find the numbers in the equation, is one that can never be misinterpreted. It is unnecessary in such an equation to state what units are used. Each person can choose the units he happens to be familiar with, or to prefer; the equation will still be correct. Such an equation is said to be in **absolute units** (the adjective is perhaps misleading).

19. We shall now see that the use of absolute units not only eliminates confusion and misunderstanding, but also enables us to obtain *a priori* information of great value. Let us take, *e.g.*, the problem of the uniform motion of a particle along the circumference of a circle. It is required to find what is the acceleration of the particle, *i.e.* what is the change in the velocity each second. Let the radius of the circle be r, which means that r is the number representing the ratio of the radius to a certain unit length; and let v be the speed, *i.e.* v is the number representing how many times a certain unit of speed has to be taken to get the speed of the particle. Only the speed and the radius can enter into the expression for the acceleration. We, therefore, wish to find what function of v and r must be taken in order to get the required acceleration, *i.e.* the number of units of acceleration contained in the required acceleration. Let a be this number: then we have

$$a=f(v, r), \qquad \qquad (1)$$

where f is at present an unknown functional form.

We have to consider the changes that are possible in the units. We have units of distance, speed, and acceleration. Now ordinary usage will make us realise that the unit of speed is best expressed as so many units of length in so many units of time; whilst the unit of acceleration is so many units of speed in so many units of time, *i.e.* in so many units of time a certain change of speed is produced, represented by so many units of length in so many units of time. We thus see that the numbers a, v, r all depend on the units of length and time.

Let us now suppose that the equation (1) is to be true for all units, and let us further imagine that the units are changed in some arbitrary manner. Thus let the unit of length be made $1/L$ times as great, the unit of time $1/T$ times as great, and let a', v', r' be the new numbers. Then

$$\begin{aligned} r' &= \text{number of new units of length in a given length } r \\ &= \text{old number multiplied by } L \\ &= rL; \end{aligned}$$

v'=number of new units of speed in a given speed v
=number of new units of length in a time $1/T$ times as great
=old number multiplied by L, divided by T
$=vL/T$; and
a'=number of units of change of speed in a given acceleration a
=number of new units of speed in a time $1/T$ times as great
$=1/T$ multiplied by number of new units of length in a time $1/T$ times as great
$=1/T$ multiplied by old number, divided by T, multiplied by L
$=aL/T^2$.

If, now, the equation $a=f(v, r)$ is to be true always we must have

$$a=f(v, r); \quad a'=f(v', r');$$

i.e.

$$a=f(v, r); \quad aL/T^2=f(vL/T, rL).$$

Thus

$$L/T^2 . f(v, r)=f(vL/T, rL),$$

at once suggesting that $f(v, r)$ must contain v as a factor v^2, since in $f(vL/T, rL)$, T occurs only in the first argument vL/T. Put then

$$f(v, r)=v^2\phi(r),$$

we get

$$L/T^2 . v^2\phi(r)=(vL/T)^2\phi(rL),$$

i.e.

$$1/L . \phi(r)=\phi(rL),$$

showing that $\phi(r)$ is proportional to $1/r$. Hence we deduce that

$$a \text{ is proportional to } v^2/r, \quad \dots\dots\dots\dots \quad (2)$$

or, in other terms, the number of units of acceleration is proportional to the result of dividing the square of the number of units of velocity by the number of units of length in the radius. The constant of proportionality is not given by this method, but the nature of the acceleration is indicated.

The use of this method is dependent upon care being taken to consider *all* the units that enter into any given expression. Thus, if by mistake it is supposed that the acceleration depends only on the velocity and not on the radius, we should get

$$a=f(v), \quad a'=f(v'),$$

so that

$$a=f(v), \quad aL/T^2=f(vL/T),$$

giving

$$L/T^2 . f(v)=f(vL/T),$$

which is impossible.

20. **The Fundamental Concepts of Mechanics.**—We have to consider, therefore, what units do occur in physical phenomena. It has been the experience of ages that in mechanical ideas there enter three fundamental conceptions, viz. **mass, length,** and **time.** All mechanical quantities are defined if we define the units of these three conceptions. Thus a velocity is obtained by considering a length described in a given time, so that the number of units of velocity is the number of units of length divided by the number of units of time. A force is defined as the mass of body multiplied by the rate of change of velocity,

i.e. the number of units of force is given by the number of units of mass multiplied by the number of units of acceleration; whilst the number of units of acceleration is defined by the number of units of velocity divided by the number of units of time in which this change of velocity takes place; whereas the velocity itself is defined by the number of units of length divided by the number of units of time required to describe this length.

In this way we can consider what change is produced in the number representing any mechanical concept by changes in the units used, viz. the fundamental units of length, mass, and time.

It thus follows, *a priori*, that an equation can generally be obtained between any physical concept and the factors that enter into the concept, if we write down the conditions that this equation shall be true independently of the system of units employed, whether foot, pound, second (f.p.s.), or centimetre, gram, second (c.g.s.), or any other set chosen for any particular purpose.

But in the application of this method care must be taken not only to include all the factors, but also to measure each factor in the set of units chosen once for all. Thus it will not do to use feet in measuring velocities, and miles in measuring distances; pounds in measuring forces, and tons in measuring weights.

If due care is taken much useful information can be obtained.

21. **Uniformly Accelerated Motion from Rest.**—Thus, what is the relation between the distance, time, and acceleration in uniformly accelerated motion, starting from rest?

The distance described must depend on the acceleration and the time. Let

s = number of units of length in the distance described;
a = number of units of acceleration; and
t = number of units of time.

Then

$$s = f(a, t) \quad \ldots\ldots\ldots\ldots \quad (3)$$

Let us suppose the unit of length diminished in ratio $1/L$, the unit of time in ratio $1/T$. Then, if s', a', t' are the new numbers representing the same physical quantities originally given by s, a, t, we have

$$s' = Ls, \quad a' = La/T^2, \quad t' = Tt.$$

But if the original equation is correct, and the units used in the first representation and in the second are each consistent in themselves, we have

$$s' = f(a', t');$$

hence we have

$$s = f(a, t), \text{ and } Ls = f(La/T^2, Tt),$$

so that

$$Lf(a, t) = f(La/T^2, Tt)$$

for all values of L and T. Since L is a factor on the left, it must be a factor on the right. Hence

$$f(a, t) = a\phi(t),$$

giving

$$La\phi(t) = La/T^2 . \phi(Tt),$$

i.e.

$$T^2\phi(t) = \phi(Tt),$$

so that $\phi(t)$ is proportional to t^2. Hence we have

$$s \text{ proportional to } at^2 \text{ (in fact, } \tfrac{1}{2}at^2) \quad \ldots\ldots\ldots \quad (4)$$

22. **Initial Velocity not Zero.**—Sometimes the information given by this method is not so complete, so that we cannot get the functional relation in full without any ambiguity. Take, *e.g.*, the acceleration problem in which the body starts off with some initial velocity in the same direction as the acceleration. This velocity now enters into the value of the distance described. If u is the number of units of velocity initially under the first system of units, and u' under the new system, then, in addition to the relations $s' = Ls$, $a' = La/T^2$, $t' = Tt$, we have $u' = Lu/T$. We now have

$$s = f(u, a, t) \text{ and } s' = f(u', a', t'), \quad \ldots \quad (5)$$

giving

$$s = f(u, a, t), \; Ls = f(Lu/T, La/T^2, Tt),$$

so that

$$Lf(u, a, t) = f(Lu/T, La/T^2, Tt).$$

We cannot now pronounce definitely on the form of f. Some information may, however, be obtained. We see that

$$f(Lu/T, La/T^2, Tt)$$

must contain a factor L. Hence* it is a homogeneous function of the first degree in Lu/T and La/T^2, i.e. $f(u, a, t)$ is a homogeneous function of the first degree in u, a. Thus we can write, for instance,

$$s = f(u, a, t) = u\phi(a/u, t); \; s' = u'\phi(a'/u', t').$$

Using the same method as before, we get

$$s = u\phi(a/u, t); \; Ls = Lu/T \,.\, \phi(a/uT, Tt),$$

giving

$$T\phi(a/u, t) = \phi(a/uT, Tt).$$

If T is to be a factor of $\phi(a/uT, Tt)$, we must have*

$$\phi(a/uT, Tt) = Tt\psi(a/uT \,.\, Tt) = Tt\psi(at/u).$$

Hence we get

$$\phi(a/u, t) = t\psi(at/u),$$

so that finally

$$s = ut\psi(at/u), \quad \ldots \quad (6)$$

where ψ is some functional form.

23. **Quantity of Zero Dimension.**—It is interesting to note why the form of ψ cannot be discussed by the present method. The quantity at/u is quite unaffected by any change of units, since

$$\frac{a't'}{u'} = \frac{La}{T^2} \,.\, tT \,.\, \frac{1}{\frac{Lu}{T}} = \frac{at}{u}$$

for all values of L and T. Thus no information about $\psi(at/u)$ can be derived from the consideration of changes of units.

We know, in fact, that

$$\left.\begin{aligned} s &= ut + \tfrac{1}{2}at^2; \\ \text{this shows that} \quad \psi(at/u) &\equiv 1 + \tfrac{1}{2}(at/u), \end{aligned}\right\} \quad \ldots \quad (7)$$

but this result must depend on other considerations, as detailed in the books on dynamics.

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* *N.B.*

24. **Pendulum.**—Again, consider the problem of the simple pendulum. A particle is tied to a weightless string; to find the time of an oscillation under the force of gravity.

The factors that affect the time of oscillation can only be the mass of the particle, the length of the string, the angle of oscillation, and the pull of gravity. Let

m = number of units of mass in the particle;
l = number of units of length in the string;
g = number of units of acceleration caused by gravity;
a = number of units of angle in half the oscillation (*i.e.* on each side of vertical);
t = number of units of time in an oscillation;

and let

$$t = f(m, l, g, a) \quad \ldots\ldots\ldots\ldots \quad (8)$$

be an equation true for any system of units. Let the units of length, mass, and time be changed in ratios $1/L$, $1/M$, $1/T$. Then the new numbers are given by

$$m' = Mm, \quad l' = Ll, \quad g' = Lg/T^2, \quad t' = Tt,$$

but $a' = a$, *i.e.* the number of units of angle is unaffected. We have here another case of a quantity unaffected by change of units, because, if an angle is measured in radians, no change is produced in the number representing the angle by any change in the unit of length (the other units are clearly irrelevant). We therefore have

$$t = f(m, l, g, a); \quad t' = f(m', l', g', a'), \text{ i.e. } Tt = f(Mm, Ll, Lg/T^2, a),$$

giving

$$Tf(m, l, g, a) = f(Mm, Ll, Lg/T^2, a)$$

for all values of L, M, T. It is at once obvious that m cannot enter into the equation, since $f(Mm, Ll, Lg/T^2, a)$ is unaffected by the value of M. Thus we can use $t = f(l, g, a)$, where

$$Tf(l, g, a) = f(Ll, Lg/T^2, a).$$

Since T only exists in the middle argument on the right-hand side, we can write

$$f(Ll, Lg/T^2, a) = (Lg/T^2)^{-\frac{1}{2}}\phi(Ll, a),$$

so that

$$t = f(l, g, a) = g^{-\frac{1}{2}}\phi(l, a).$$

This gives

$$t' = Tt = Tg^{-\frac{1}{2}}L^{-\frac{1}{2}}\phi(Ll, a),$$

so that

$$L^{\frac{1}{2}}\phi(l, a) = \phi(Ll, a).$$

We at once get

$$\phi(Ll, a) = (Ll)^{\frac{1}{2}}\psi(a),$$

so that

$$t = (l/g)^{\frac{1}{2}}\psi(a) \quad \ldots\ldots\ldots\ldots \quad (9)$$

We find that no information concerning ψ is possible by the present method.

But if we can assume that the time of oscillation is independent of the angle of oscillation—which, as shown by Galileo's experiment, is the case when the latter is small enough—we get

$$t \text{ is proportional to } (l/g)^{\frac{1}{2}},$$

since $\psi(a)$ must now be a mere constant.

25. The method of units (or dimensions) should now be clear. We imagine the units of length, mass, time changed in ratios $1/L$, $1/M$, $1/T$; find the new numbers for the various physical quantities, and consider the conditions that the substitution of these new numbers in the assumed equation leaves the equation correct. It is, of course, useful to be able to get the new numbers quickly. A table of such changes is, therefore, given for the well-known mechanical concepts :—

Quantity.	Factor of Change when Units of Length, Mass, and Time are Changed in Ratios $1/L, 1/M, 1/T$.
Length	L
Mass	M
Time	T
Angle (in Radians)	No change
Speed or Velocity	L/T
Angular Velocity (in Radians)	$1/T$
Change of Speed or Acceleration	L/T^2
Force (Mass $\times$ Change of Velocity)	ML/T^2
Moment (Force $\times$ Distance)	ML^2/T^2
Density (Mass $\div$ Volume)	M/L^3
Viscosity (see § 27)	M/LT
Kinematic Viscosity (see § 29)	L^2/T
Elasticity (see § 32)	M/LT^2

26. **Theory of Similitude — Compound Pendulum.**—It is sometimes not necessary or not possible to take into account *all* the circumstances that should enter into a physical equation. Take, *e.g.*, the case of the compound pendulum. As is well known, the time of oscillation must also depend on the shape of the pendulum and the arrangement of the matter. We can, nevertheless, dispense with this knowledge, and discover something about the time of oscillation.

Imagine two exactly similar pendulums, but one of them a certain number of times as big as the other. To fix our ideas, consider two uniform thin rods of lengths $2a$, $2na$, and let the axis of oscillation in the first be at distance h from the centre, and in the other at distance nh, Fig. 7 If we wish to find the way in which the time of oscillation is affected by the factor n, *i.e.* in what way the times differ in the two cases, we can proceed as follows.

It has already been seen in the case of the simple pendulum that the total mass cannot affect the time of oscillation. But, in any case, to be on the safe side, let us suppose that the pendulum a has a mass m_1. Then, using a notation similar to that in § 24, we have

$$t_1 = f(m_1, a, h, g, a_1), \qquad \qquad (10)$$

where a_1 is the half-amplitude of the first pendulum. Also

$$t_1' = Tt_1,\ m_1' = Mm_1,\ a' = La,\ h' = Lh,\ g' = Lg/T^2,\ a_1' = a_1;$$

we get

$$Tt_1 = f(Mm_1, La, Lh, Lg/T^2, a_1),$$

so that

$$Tf(m_1, a, h, g, a_1) = f(Mm_1, La, Lh, Lg/T^2, a_1).$$

2*

Hence, as before, f does not involve m_1. Writing

$$t_1 = f(a, h, g, a_1),$$

we get

$$Tf(a, h, g, a_1) = f(La, Lh, Lg/T^2, a_1),$$

so that we can put

$$f(a, h, g, a_1) = g^{-\frac{1}{2}}\phi(a, h, a_1).$$

We deduce

$$L^{\frac{1}{2}}\phi(a, h, a_1) = \phi(La, Lh, a_1).$$

Thus $\phi(La, Lh, a_1)$ must have a factor $L^{\frac{1}{2}}$, so that we can write

$$\phi(La, Lh, a_1) = (La)^{\frac{1}{2}}\psi(Lh/La, a_1) = (La)^{\frac{1}{2}}\psi(h/a, a_1),$$

giving

$$t_1 = (a/g)^{\frac{1}{2}}\psi(h/a, a_1).$$

Here we cannot go any further, since h/a and a_1 are both unaffected by changes in units.

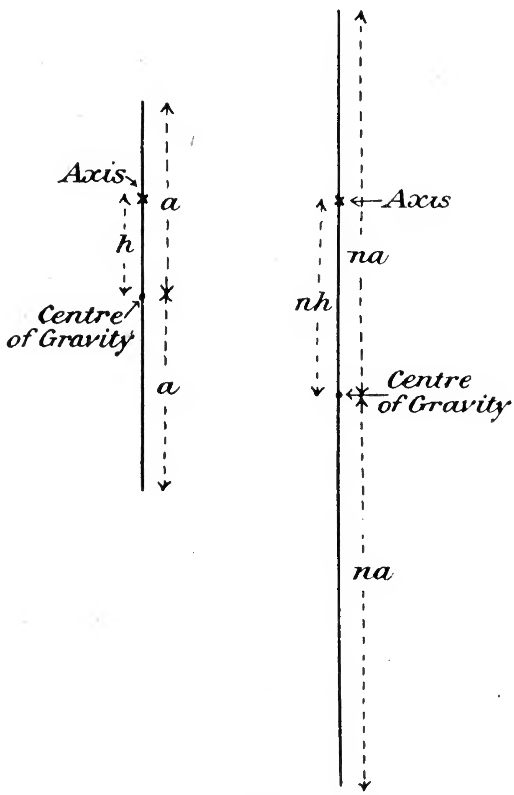

FIG. 7.—Theory of Similitude: Pendulum.

But we see that in the second pendulum, where we have na and nh instead of a and h, we must have

$$t_2 = (na/g)^{\frac{1}{2}}\psi(h/a, a_2), \quad \ldots \ldots \quad (11)$$

where t_2 is the period of the second pendulum with angle of oscillation a_2.

If $a_1 = a_2$, *i.e.* both pendulums have the same amplitude, we get

$$t_2 = n^{\frac{1}{2}} t_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We thus conclude that exactly similar pendulums have times of oscillation as the square root of the ratio of the corresponding sizes.

27. **Air Resistance—Sphere.**—We are now in a position to consider the theory of units in relation to the question of air resistance. Let us begin with a simple case.

A sphere moves without rotation in an incompressible fluid "at rest." To discuss the fluid resistance we must discover, first, *all* the circumstances that affect the result *in the case of spheres*. They are (1) the size of the sphere, (2) the velocity, (3) the density of the fluid, (4) the viscosity of the fluid. We cannot think of other physical quantities that can affect the resistance. If any such exist, we may discover this in the end by getting impossible equations. Let

r = number of units of length in the radius of the sphere;
U = number of units of velocity;
ρ = number of units of mass per unit volume of the air;
μ = number of units of viscosity.

We must discuss the quantity μ in order to find out how it is affected by change of units. The definition of the coefficient of viscosity is (Maxwell):

> "*The viscosity of a substance is measured by the tangential force on unit area of either of two horizontal planes at unit distance apart, one of which is fixed, while the other moves with unit velocity, the space between being filled with the viscous substance.*"

The quantity μ is measured by a force per unit area, and varies inversely as the velocity per unit distance of separation for a given force per unit area. Suppose, then, that a certain force over a certain area is obtained when the distance between the planes is given and the velocity is given. With the change of units already suggested, we find that the force becomes ML/T^2 as great, the area L^2 as great, the distance L times as great, and the velocity L/T as great. Hence the new value of μ, called μ', is given by

$$\mu' = \left(\mu \frac{ML}{T^2} \cdot \frac{1}{L^2}\right) \Big/ \left(\frac{L}{T} \cdot \frac{1}{L}\right) = \frac{M}{LT}\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Also $r' = Lr$, $U' = LU/T$, $\rho' = M\rho/L^3$.

Let us now write for the air resistance in the case of spheres

$$R = f(r, U, \rho, \mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

R is a force, so that with a change of units we get $R' = \dfrac{ML}{T^2} R$.

If, now, the proposed equation is to be true for any system of units, we have

$$R' = f(r', U', \rho', \mu'),$$

i.e.

$$\frac{ML}{T^2} R = f\left(Lr, \frac{LU}{T}, \frac{M\rho}{L^3}, \frac{M\mu}{LT}\right),$$

so that

$$\frac{ML}{T^2} f(r, U, \rho, \mu) = f\left(Lr, \frac{LU}{T}, \frac{M\rho}{L^3}, \frac{M\mu}{LT}\right) \quad . \quad . \quad . \quad . \quad (15$$

28. **No Viscosity.**—Suppose that the viscosity does not really affect the resistance, as, *e.g.*, imagine that the fluid is a perfect fluid. Then μ does not enter into the equation, and we have

$$\frac{ML}{T^2}f(r, U, \rho)=f\left(Lr, \frac{LU}{T}, \frac{M\rho}{L^3}\right).$$

To get a factor M on the right-hand side, we, therefore, write

$$f(r, U, \rho)=\rho\phi(r, U),$$

which, after a little reduction, gives

$$\frac{L^4}{T^2}\phi(r, U)=\phi\left(Lr, \frac{LU}{T}\right).$$

Hence ϕ must contain a factor U^2, say $\phi(r, U) = U^2\psi(r)$, so that

$$L^2\psi(r)=\psi(Lr),$$

and finally $\psi(r)$ varies as r^2.

Hence we deduce that when fluid resistance is independent of the viscosity, *then in the case of spheres* we have

$$R \text{ proportional to } \rho r^2 U^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Thus the resistance varies as the density, the square of the radius (i.e. *the surface of the sphere*), *and the square of the velocity.* The constant of proportionality must be obtained otherwise, *e.g.* by experiment as an approximation or by hydrodynamical theory.

29. **Viscosity not Vanishingly Small.**—Now let us take into account the possibility of the resistance being affected by the viscosity. Returning to the equation (15) above, we at once suggest that

$$f(r, U, \rho, \mu)=\rho\phi\left(r, U, \frac{\mu}{\rho}\right),$$

since both ρ and μ give factors M in ρ', μ', and the right-hand side in (15) must have a factor M. We find that

$$\frac{L^4}{T^2}\phi\left(r, U, \frac{\mu}{\rho}\right)=\phi\left(Lr, \frac{LU}{T}, \frac{L^2}{T}\frac{\mu}{\rho}\right).$$

To get a factor L^4 we now put

$$\phi\left(r, U, \frac{\mu}{\rho}\right)=r^2U^2\psi\left(\frac{\mu}{\rho Ur}\right),$$

so that we obtain

$$\frac{L^4}{T^2}r^2U^2\psi\left(\frac{\mu}{\rho Ur}\right)=L^2r^2\frac{L^2U^2}{T^2}\psi\left(\frac{L^2}{T}\frac{\mu}{\rho}\frac{T}{L^2rU}\right),$$

i.e.

$$\psi\left(\frac{\mu}{\rho Ur}\right)=\psi\left(\frac{\mu}{\rho Ur}\right),$$

an identity. This is because $\mu/\rho Ur$ is unchanged by change of units. ψ cannot, therefore, be evaluated further by the theory of dimensions.

The quantity μ/ρ is called the **kinematic viscosity** and is denoted by ν. We, therefore, obtain

$$R=\rho r^2U^2\psi\left(\frac{\nu}{Ur}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

30. It may be desirable, for certain purposes, to discover the form of ψ. But for many purposes this is not necessary. Thus, if we arrange two spheres of radii r_1, r_2, moving in the same fluid with velocities U_1, U_2, in such a way that

$$U_1 r_1 = U_2 r_2,$$

then

$$R_1 = \rho r_1^2 U_1^2 \psi\left(\frac{\nu}{U_1 r_1}\right), \quad R_2 = \rho r_2^2 U_2^2 \psi\left(\frac{\nu}{U_2 r_2}\right),$$

i.e. $R_1 = R_2$. We can, therefore, say that spheres have the same resistance if the velocities are inversely proportional to the radii.

Or suppose we wish to find the resistances for spheres in air by means of experiments in water. (The results here proved are true for any fluid.) Let a sphere radius r_1 move with velocity U_1 in air of kinematic viscosity ν_1 and density ρ_1. Choose a sphere of radius r_2 to move in water, density ρ_2, kinematic viscosity ν_2, with such a velocity U_2 that

$$\frac{\nu_2}{U_2 r_2} = \frac{\nu_1}{U_1 r_1}.$$

Then

$$R_1 = \rho_1 r_1^2 U_1^2 \psi\left(\frac{\nu_1}{U_1 r_1}\right), \quad R_2 = \rho_2 r_2^2 U_2^2 \psi\left(\frac{\nu_2}{U_2 r_2}\right),$$

so that

$$\frac{R_1}{R_2} = \frac{\rho_1}{\rho_2}\left(\frac{r_1 U_1}{r_2 U_2}\right)^2 = \frac{\rho_1}{\rho_2}\left(\frac{\nu_1}{\nu_2}\right)^2.$$

Thus the required resistance can be readily calculated from the experimental results.

31. **General Case, with No Rotation.**—Let us now proceed to the more generalised problem. Imagine a body of *given shape* whose size is determined by means of a certain length. For instance, in an aeroplane of *given design* let the size be determined by, say, the span of the wings (or the end-to-end length of the body, or any other such well-defined length). It will be our object to discover how this length enters into the expression for air resistance.

Assume that *for the given design* the formula is

$$R = f(l, U, \rho, \mu), \quad \text{(18)}$$

where l is the length determining the size, and U, ρ, μ are as before, U being along a direction fixed with respect to the body. By means of exactly the same argument as in § 28, we deduce that if the viscosity is of no effect on the resistance, then

$$R \text{ is proportional to } \rho l^2 U^2, \quad \text{(16)}$$

but that if the viscosity does enter into the value of the resistance, then

$$R = \rho l^2 U^2 \psi\left(\frac{\nu}{Ul}\right), \quad \text{(17)}$$

where ψ is some function whose form cannot be given by the method of dimensions.

If it is found that in fact R is proportional to $\rho l^2 U^2$, then $\psi(\nu/lU)$ must be a constant, which means that the viscosity does not affect the resistance.

32. **Elasticity.**—We have assumed that the air is incompressible. This is true enough for water, but in the case of air, especially in view of

the high velocities now usual in flight, it is by no means obvious *a priori* that the elasticity of the air should not affect the resistance. In a recent volume on air resistance* arguments are advanced to prove that, in fact, the elasticity of air plays some part in the production of resistance. In the practical application of the theory it is usual to ignore this effect. It will, nevertheless, be of theoretical interest to examine the way in which the elasticity affects the resistance.

We have first to consider how to represent the elasticity numerically. The coefficient of elasticity of a gas is defined (Maxwell) as:

"*The ratio of any small increase of pressure to the voluminal compression hereby produced.*"

Now the voluminal compression, *i.e.* compression per unit volume, is the ratio of a change of volume to the original volume: it is, therefore, the same for all units. Hence by a change of units the new coefficient of elasticity is the old coefficient multiplied by the same ratio as for a pressure, *i.e.* for a force per unit area. Thus the multiplying factor is $\frac{ML}{T^2}\Big/L^2$, *i.e.* $\frac{M}{LT^2}$, so that if E is the coefficient with a system of units, then the new coefficient E' obtained by the change considered is given by

$$E' = \frac{M}{LT^2}E \,. \qquad (19)$$

Using this result, we can find the elasticity effect in the manner already exemplified.

33. **Elasticity and the Velocity of Sound.**—In practice this would not be very convenient. Another method is adopted which depends upon the fact that the elasticity of a gas enters into the velocity of sound in the gas. Thus, instead of E, we can use the velocity of sound in the fluid. Let V be this velocity. A change of units gives

$$V' = \frac{LV}{T}.$$

Let us then write

$$R = f(l, U, \rho, \mu, V), \qquad (20)$$

where l, U, ρ, μ are the same as before. If this equation is true for all units, we have

$$R' = f(l', U', \rho', \mu', V'),$$

where

$$R' = \frac{ML}{T^2}R,\ l' = Ll,\ U' = \frac{LU}{T},\ \rho' = \frac{M\rho}{L^3},\ \mu' = \frac{M\mu}{LT},\ V' = \frac{LV}{T}.$$

Hence

$$\frac{ML}{T^2}f(l, U, \rho, \mu, V) = f\left(Ll, \frac{LU}{T}, \frac{M\rho}{L^3}, \frac{M\mu}{LT}, \frac{LV}{T}\right).$$

Using the same arguments as before, we deduce

$$f\left(l, U, \rho, \mu, V\right) = \rho\phi\left(l, U, \frac{\mu}{\rho}, V\right),$$

giving

$$\frac{L^4}{T^2}\phi\left(l, U, \frac{\mu}{\rho}, V\right) = \phi\left(Ll, \frac{LU}{T}, \frac{L^2\mu}{T\rho}, \frac{LV}{T}\right).$$

* *Resistance of Air*, by Lt.-Col. R. de Villamil.

We now write

$$\phi\left(l, U, \frac{\mu}{\rho}, V\right) = U^2\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}\right),$$

so as to account for the T^2 in the denominator; we obtain

$$L^2\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}\right) = \psi\left(Ll, \frac{L\mu}{\rho U}, \frac{V}{U}\right),$$

so that finally

$$\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}\right) = l^2\chi\left(\frac{\mu}{\rho Ul}, \frac{V}{U}\right).$$

The function χ cannot be reduced any further (it depends on the shape, direction of motion, etc.): the quantities $\mu/\rho Ul$, V/U are independent of units. We thus find

$$R = \rho l^2 U^2 \chi\left(\frac{\nu}{lU}, \frac{V}{U}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

We see that just as the viscosity enters into R in a function involving the ratio ν/lU, so the elasticity enters by means of R involving the ratio U/V. Thus the question whether or not the elasticity is important is reduced to the question whether the velocity of the body is comparable to that of sound in the medium under consideration. It is the generally accepted belief that for ordinary aeroplane speeds, say to 200 ft. per second, the elasticity is of no importance.

34. **Rotation Included.**—Let us now suppose that the body, in addition to its velocity U along a direction fixed in the body, has an angular velocity Ω about some axis fixed in the body. And let us consider a form for the resistance which shall apply to all bodies of the same shape with translational motion in the same relative direction, and rotational motion about an axis also in a fixed relative direction. On the same assumptions as have been made before, we have

$$R = f(l, U, \rho, \mu, V, \Omega), \quad R' = f(l', U', \rho', \mu', V', \Omega'), \quad . \quad . \quad . \quad (22)$$

where l' in terms of l, etc., are the same as before, and $\Omega' = \Omega/T$, as the student can verify by easy argument. The process of reasoning is indicated by the following:—

$$\frac{ML}{T^2} f\left(l, U, \rho, \mu, V, \Omega\right) = f\left(Ll, \frac{LU}{T}, \frac{M\rho}{L^3}, \frac{M\mu}{LT}, \frac{LV}{T}, \frac{\Omega}{T}\right),$$

therefore

$$f\left(l, U, \rho, \mu, V, \Omega\right) = \rho\phi\left(l, U, \frac{\mu}{\rho}, V, \Omega\right);$$

$$\frac{L^4}{T^2}\phi\left(l, U, \frac{\mu}{\rho}, V, \Omega\right) = \phi\left(Ll, \frac{LU}{T}, \frac{L^2\mu}{T\rho}, \frac{LV}{T}, \frac{\Omega}{T}\right),$$

therefore

$$\phi\left(l, U, \frac{\mu}{\rho}, V, \Omega\right) = U^2\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}, \frac{\Omega}{U}\right);$$

$$L^2\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}, \frac{\Omega}{U}\right) = \psi\left(Ll, \frac{L\mu}{\rho U}, \frac{V}{U}, \frac{\Omega}{LU}\right),$$

therefore

$$\psi\left(l, \frac{\mu}{\rho U}, \frac{V}{U}, \frac{\Omega}{U}\right) = l^2\chi\left(\frac{\mu}{l\rho U}, \frac{V}{U}, \frac{l\Omega}{U}\right);$$

so that

$$R = \rho l^2 U^2 \chi\left(\frac{\nu}{lU}, \frac{V}{U}, \frac{l\Omega}{U}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The three arguments in χ are unaffected by changes of units. Thus we cannot go any further by the present method. As the result stands it can still be of use to indicate important facts. Since the rotation enters into R in the form of $l\Omega/U$, we see that for exactly similar cases the angular motion is more important the greater the body and the less the velocity U.

35. **General Problem (Elasticity Neglected).**—We have, so far, only considered the resultant resistance R. Its position and direction are, of course, fixed relatively to the body in each case we have examined. For the sphere R evidently passes through the centre, unless rotation exists. In general R will not necessarily pass through any particularly chosen point. It is clear, however, that the moment of R about any axis fixed relatively to the body is proportional to lR.

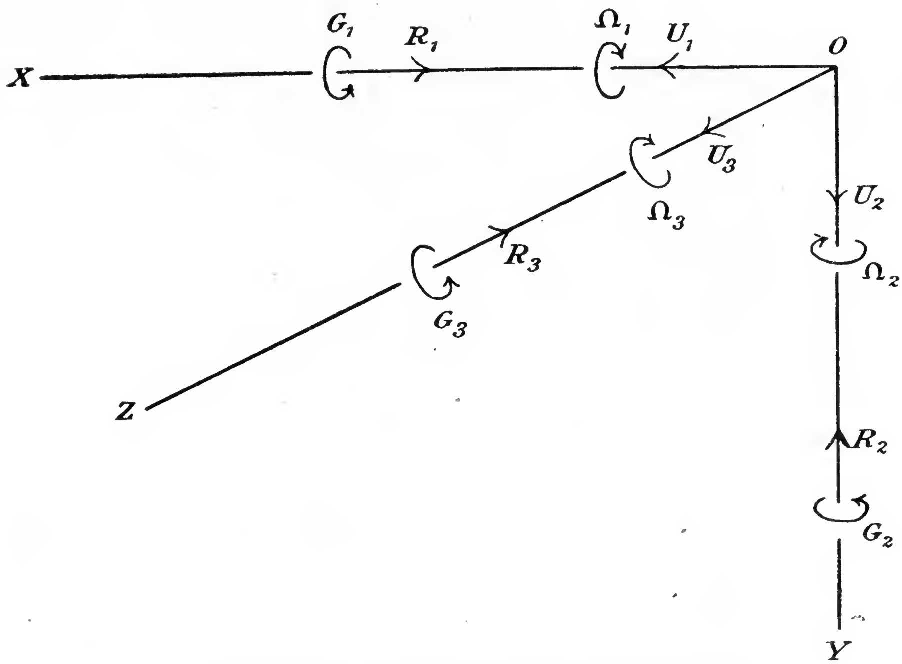

FIG. 8.—Body Axes: Velocity and Resistance Components.

But we wish to go further and consider not merely the value of R, but also the components of R and the moments about given axes. Suppose also that the direction of U is not given relatively to the body, and the axis of Ω is also not so given. In other words, we wish to consider the general problem of resistance for a body moving steadily in any manner in a fluid.

For this purpose it is necessary to define axes of reference which are fixed in the body, and with respect to which translational and angular velocities are measured. We shall in the main argument of this book use the axes shown in Fig. 8. O is some given point in the body (usually the centre of gravity), and OX, OY, OZ are three lines, mutually perpendicular, which pass through O and form a right-handed system of axes. Let the velocity of translation have components U_1, U_2, U_3 *along* the axes of

X, Y, Z; and let the angular velocity have components $\Omega_1, \Omega_2, \Omega_3$ *about* the axes of X, Y, Z (see Chapter IV., § 99). The senses of $\Omega_1, \Omega_2, \Omega_3$ are such that each rotation bears to the corresponding translation the relation of a right-handed screw.

The effect of the air can be represented by the vectorial sum of a number of resistances on the elements of surface of the body. We, therefore, get in general three components of resistance force, R_1, R_2, R_3, which we shall measure positively *along the negative directions* of X, Y, Z; and three components of resistance couple, G_1, G_2, G_3, which we shall measure positively *in the negative senses* of $\Omega_1, \Omega_2, \Omega_3$. The conventions adopted are illustrated in Fig. 8, the arrow associated with each quantity being indicative of the positive sense of this quantity.

Take R_1 as a typical component of resistance force. We have

$$R_1 = f_x(l, \rho, \mu, V, U_1, U_2, U_3, \Omega_1, \Omega_2, \Omega_3), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where l, ρ, μ, V are as defined above. Hence

$$\frac{ML}{T^2} f_x\left(l, \rho, \mu, V, U_1, U_2, U_3, \Omega_1, \Omega_2, \Omega_3\right)$$
$$= f_x\left(Ll, \frac{M\rho}{L^3}, \frac{M\mu}{LT}, \frac{LV}{T}, \frac{LU_1}{T}, \frac{LU_2}{T}, \frac{LU_3}{T}, \frac{\Omega_1}{T}, \frac{\Omega_2}{T}, \frac{\Omega_3}{T}\right),$$

so that

$$f_x = \rho\phi_x\left(l, \frac{\mu}{\rho}, V, U_1, U_2, U_3, \Omega_1, \Omega_2, \Omega_3\right).$$

Thus

$$\frac{L^4}{T^2}\phi_x\left(l, \frac{\mu}{\rho}, V, U_1, U_2, U_3, \Omega_1, \Omega_2, \Omega_3\right)$$
$$= \phi_x\left(Ll, \frac{L^2\mu}{T\rho}, \frac{LV}{T}, \frac{LU_1}{T}, \frac{LU_2}{T}, \frac{LU_3}{T}, \frac{\Omega_1}{T}, \frac{\Omega_2}{T}, \frac{\Omega_3}{T}\right).$$

Let U be the resultant velocity, so that

$$U = (U_1^2 + U_2^2 + U_3^2)^{\frac{1}{2}}.$$

Then

$$\phi_x = U^2\psi_x\left(l, \frac{\mu}{\rho U}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{\Omega_1}{U}, \frac{\Omega_2}{U}, \frac{\Omega_3}{U}\right)$$

in order to get a factor T^2. Hence

$$L^2\psi_x\left(l, \frac{\mu}{\rho U}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{\Omega_1}{U}, \frac{\Omega_2}{U}, \frac{\Omega_3}{U}\right)$$
$$= \psi_x\left(Ll, \frac{L\mu}{\rho U}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{\Omega_1}{LU}, \frac{\Omega_2}{LU}, \frac{\Omega_3}{LU}\right).$$

This gives

$$\psi_x = l^2\chi_x\left(\frac{\mu}{l\rho U}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right),$$

so that

$$R_1 = \rho l^2 U^2\chi_x\left(\frac{\nu}{lU}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right) \quad . \quad . \quad . \quad . \quad (25)$$

36. **Predominating Velocity Component.**—Instead of the resultant velocity U we can use any other function of U_1, U_2, U_3 having

the same dimensions as U. Thus we may use U_1 itself, or U_2, or U_3, or $(U_1U_2)^{\frac{1}{2}}$, etc. If we choose U_1, *e.g.*, we get

$$R_1 = \rho l^2 U_1{}^2 \chi_x\left(\frac{\nu}{lU_1}, \frac{V}{U_1}, \frac{U_2}{U_1}, \frac{U_3}{U_1}, \frac{l\Omega_1}{U_1}, \frac{l\Omega_2}{U_1}, \frac{l\Omega_3}{U_1}\right), \quad . \quad . \quad . \quad . \quad (26)$$

where χ_x is now a different function of the arguments indicated in the brackets. We note that all the arguments in (25) and (26) are of zero dimensions, *i.e.* they are unaffected by any change of units. Hence the forms of the functions in these expressions can only be determined by aerodynamical or experimental methods.

We can at once write

$$R_2 = \rho l^2 U^2 \chi_y\left(\frac{\nu}{lU}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right),$$

$$R_3 = \rho l^2 U^2 \chi_z\left(\frac{\nu}{lU}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right),$$

with the notation of § 35 ; or

$$R_2 = \rho l^2 U_1{}^2 \chi_y\left(\frac{\nu}{lU_1}, \frac{V}{U_1}, \frac{U_2}{U_1}, \frac{U_3}{U_1}, \frac{l\Omega_1}{U_1}, \frac{l\Omega_2}{U_1}, \frac{l\Omega_3}{U_1}\right),$$

$$R_3 = \rho l^2 U_1{}^2 \chi_z\left(\frac{\nu}{lU_1}, \frac{V}{U_1}, \frac{U_2}{U_1}, \frac{U_3}{U_1}, \frac{l\Omega_1}{U_1}, \frac{l\Omega_2}{U_1}, \frac{l\Omega_3}{U_1}\right),$$

using the velocity component U_1 instead of U. In the same way we have for the components of resistance couple

$$\left.\begin{aligned} &G_1 = \rho l^3 U^2 \chi_1\left(\frac{\nu}{lU}, \frac{V}{U}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right), \\ \text{or} \quad & \\ &G_1 = \rho l^3 U_1{}^2 \chi_1\left(\frac{\nu}{lU_1}, \frac{V}{U_1}, \frac{U_2}{U_1}, \frac{U_3}{U_1}, \frac{l\Omega_1}{U_1}, \frac{l\Omega_2}{U_1}, \frac{l\Omega_3}{U_1}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

with similar expressions for G_2, G_3.

The functions $\chi_x, \chi_y, \chi_z, \chi_1, \chi_2, \chi_3$ depend on the configuration of the body and on the position of the axes in the body. For given shape and axes they can be found (at any rate, theoretically speaking, they exist).

It will be noticed that in the form (25) for R_1 we have as arguments the ratios U_1/U, U_2/U, U_3/U. These are not all independent quantities, since the sum of their squares is unity. Thus the alternative form (26) or some similar modification is preferable from the point of view of mathematical brevity. This preference is reinforced by other considerations in practical applications. In almost all problems on flight the velocity does not vary in direction or amount by more than comparatively small quantities. There is thus *an average value and direction of the velocity.* The X axis is often chosen along this direction, and hence U_1 is the predominating component of velocity, U_2/U_1, U_3/U_1 being small fractions. Further, in practice $l\Omega_1/U_1$, $l\Omega_2/U_1$, $l\Omega_3/U_1$ are also small fractions. Hence the alternative form (26) and others that can be readily suggested are particularly suitable for some purposes of aeroplane mathematics, all the arguments due to the motion being small quantities.

37. **Derivatives for Small Changes in the Motion.**—If, now, there is a small change in the motion we can at once deduce the forms of the changes in the resistance components. For instance, let U_1 become $U_1 + u_1$, where u_1 is small. The change in R_1 is, to the first order of

small quantities, given by $u_1 \partial R_1/\partial U_1$. If changes take place simultaneously in U_2, U_3, Ω_1, Ω_2, Ω_3, represented by additional quantities u_2, u_3, ω_1, ω_2, ω_3, the new motion being assumed steady, then the change in R_1 is, to the first order of small quantities,

$$u_1\frac{\partial R_1}{\partial U_1}+u_2\frac{\partial R_1}{\partial U_2}+u_3\frac{\partial R_1}{\partial U_3}+\omega_1\frac{\partial R_1}{\partial \Omega_1}+\omega_2\frac{\partial R_1}{\partial \Omega_2}+\omega_3\frac{\partial R_1}{\partial \Omega_3}.$$

Similarly, the changes in R_2, R_3, G_1, G_2, G_3 are respectively

$$\left.\begin{aligned}
&u_1\frac{\partial R_2}{\partial U_1}+u_2\frac{\partial R_2}{\partial U_2}+u_3\frac{\partial R_2}{\partial U_3}+\omega_1\frac{\partial R_2}{\partial \Omega_1}+\omega_2\frac{\partial R_2}{\partial \Omega_2}+\omega_3\frac{\partial R_2}{\partial \Omega_3};\\
&u_1\frac{\partial R_3}{\partial U_1}+u_2\frac{\partial R_3}{\partial U_2}+u_3\frac{\partial R_3}{\partial U_3}+\omega_1\frac{\partial R_3}{\partial \Omega_1}+\omega_2\frac{\partial R_3}{\partial \Omega_2}+\omega_3\frac{\partial R_3}{\partial \Omega_3};\\
&u_1\frac{\partial G_1}{\partial U_1}+u_2\frac{\partial G_1}{\partial U_2}+u_3\frac{\partial G_1}{\partial U_3}+\omega_1\frac{\partial G_1}{\partial \Omega_1}+\omega_2\frac{\partial G_1}{\partial \Omega_2}+\omega_3\frac{\partial G_1}{\partial \Omega_3};\\
&u_1\frac{\partial G_2}{\partial U_1}+u_2\frac{\partial G_2}{\partial U_2}+u_3\frac{\partial G_2}{\partial U_3}+\omega_1\frac{\partial G_2}{\partial \Omega_1}+\omega_2\frac{\partial G_2}{\partial \Omega_2}+\omega_3\frac{\partial G_2}{\partial \Omega_3};\\
&u_1\frac{\partial G_3}{\partial U_1}+u_2\frac{\partial G_3}{\partial U_2}+u_3\frac{\partial G_3}{\partial U_3}+\omega_1\frac{\partial G_3}{\partial \Omega_1}+\omega_2\frac{\partial G_3}{\partial \Omega_2}+\omega_3\frac{\partial G_3}{\partial \Omega_3}.
\end{aligned}\right\} \quad . \quad . \quad (28)$$

Referring to the form of R_1 in (26), we have

$$\frac{\partial R_1}{\partial U_1}=2\rho l^2 U_1\chi_x+\rho l^2 U_1^2\left\{-\frac{\nu}{lU_1^2}\frac{\partial\chi_x}{\partial\left(\frac{\nu}{lU_1}\right)}-\frac{V}{U_1^2}\frac{\partial\chi_x}{\partial\left(\frac{V}{U_1}\right)}-\frac{U_2}{U_1^2}\frac{\partial\chi_x}{\partial\left(\frac{U_2}{U_1}\right)}-\ldots\right\}$$

$$=\rho l^2 U_1\left[2\chi_x-\frac{\nu}{lU_1}\frac{\partial\chi_x}{\partial\left(\frac{\nu}{lU_1}\right)}-\frac{V}{U_1}\frac{\partial\chi_x}{\partial\left(\frac{V}{U_1}\right)}-\frac{U_2}{U_1}\frac{\partial\chi_x}{\partial\left(\frac{U_2}{U_1}\right)}-\ldots\right]=\ldots$$

$$=\rho l^2 U \textit{ into a function of the arguments of } \chi_x \textit{ in } (25) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

In the same way,

$$\frac{\partial R_1}{\partial U_2}=\rho l^2 U_1^2\cdot\frac{1}{U_1}\frac{\partial\chi_x}{\partial\left(\frac{U_2}{U_1}\right)}=\ldots$$

$$=\rho l^2 U \textit{ into a function of the arguments of } \chi_x \textit{ in } (25)\ . \quad (30)$$

The same applies to all the thirty-six differential coefficients of R_1, R_2, R_3, G_1, G_2, G_3 with respect to the variables U_1, U_2, U_3, Ω_1, Ω_2, Ω_3: each *derivative* contains the factor U. We, therefore, introduce the following notation:—

$$\left.\begin{aligned}
&\frac{\partial R_1}{\partial U_1}=Ua_x,\ \frac{\partial R_1}{\partial U_2}=Ub_x,\ \frac{\partial R_1}{\partial U_3}=Uc_x,\ \frac{\partial R_1}{\partial \Omega_1}=Ud_x,\ \frac{\partial R_1}{\partial \Omega_2}=Ue_x,\ \frac{\partial R_1}{\partial \Omega_3}=Uf_x;\\
&\frac{\partial R_2}{\partial U_1}=Ua_y,\ \frac{\partial R_2}{\partial U_2}=Ub_y,\ \frac{\partial R_2}{\partial U_3}=Uc_y,\ \frac{\partial R_2}{\partial \Omega_1}=Ud_y,\ \frac{\partial R_2}{\partial \Omega_2}=Ue_y,\ \frac{\partial R_2}{\partial \Omega_3}=Uf_y;\\
&\frac{\partial R_3}{\partial U_1}=Ua_z,\ \frac{\partial R_3}{\partial U_2}=Ub_z,\ \frac{\partial R_3}{\partial U_3}=Uc_z,\ \frac{\partial R_3}{\partial \Omega_1}=Ud_z,\ \frac{\partial R_3}{\partial \Omega_2}=Ue_z,\ \frac{\partial R_3}{\partial \Omega_3}=Uf_z;\\
&\frac{\partial G_1}{\partial U_1}=Ua_1,\ \frac{\partial G_1}{\partial U_2}=Ub_1,\ \frac{\partial G_1}{\partial U_3}=Uc_1,\ \frac{\partial G_1}{\partial \Omega_1}=Ud_1,\ \frac{\partial G_1}{\partial \Omega_2}=Ue_1,\ \frac{\partial G_1}{\partial \Omega_3}=Uf_1;\\
&\frac{\partial G_2}{\partial U_1}=Ua_2,\ \frac{\partial G_2}{\partial U_2}=Ub_2,\ \frac{\partial G_2}{\partial U_3}=Uc_2,\ \frac{\partial G_2}{\partial \Omega_1}=Ud_2,\ \frac{\partial G_2}{\partial \Omega_2}=Ue_2,\ \frac{\partial G_2}{\partial \Omega_3}=Uf_2;\\
&\frac{\partial G_3}{\partial U_1}=Ua_3,\ \frac{\partial G_3}{\partial U_2}=Ub_3,\ \frac{\partial G_3}{\partial U_3}=Uc_3,\ \frac{\partial G_3}{\partial \Omega_1}=Ud_3,\ \frac{\partial G_3}{\partial \Omega_2}=Ue_3,\ \frac{\partial G_3}{\partial \Omega_3}=Uf_3.
\end{aligned}\right\} \quad (31)$$

The derivatives for the *force* components are characterised by *literal suffixes* x, y, z, whilst the derivatives for the *couple* components are characterised by the *numerical suffixes* 1, 2, 3. The changes in the components of resistance force and couple are, therefore,

$$\left.\begin{aligned}
&U(a_x u_1 + b_x u_2 + c_x u_3 + d_x \omega_1 + e_x \omega_2 + f_x \omega_3),\\
&U(a_y u_1 + b_y u_2 + c_y u_3 + d_y \omega_1 + e_y \omega_2 + f_y \omega_3),\\
&U(a_z u_1 + b_z u_2 + c_z u_3 + d_z \omega_1 + e_z \omega_2 + f_z \omega_3),\\
&U(a_1 u_1 + b_1 u_2 + c_1 u_3 + d_1 \omega_1 + e_1 \omega_2 + f_1 \omega_3),\\
&U(a_2 u_1 + b_2 u_2 + c_2 u_3 + d_2 \omega_1 + e_2 \omega_2 + f_2 \omega_3),\\
&U(a_3 u_1 + b_3 u_2 + c_3 u_3 + d_3 \omega_1 + e_3 \omega_2 + f_3 \omega_3),
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where the quantities a_x, b_x, c_x, d_x, e_x, f_x, etc., are all obtained by multiplying ρl^2 into functions of the arguments

$$\frac{\nu}{lU}, \ \frac{V}{U}, \ \frac{U_1}{U}, \ \frac{U_2}{U}, \ \frac{U_3}{U}, \ \frac{l\Omega_1}{U}, \ \frac{l\Omega_2}{U}, \ \frac{l\Omega_3}{U}.$$

The quantities a, b, c, d, e, f are called *Resistance Derivatives*, since they define the rates of change of the resistance forces and couples with changing steady motion of the body. The derivatives that depend on the rotations, *i.e.* d, e, f, are sometimes referred to under the title *Rotary Derivatives*. If there is given steady motion the derivatives are definite quantities dependent on the steady motion and the configuration of the body and the axes of reference. They can thus be looked upon as determinate or determinable quantities. In this book we take definite axes in the body once for all, and refer all motions to them.

38. As has already been remarked, theoretical hydrodynamical methods are at present insufficient to discover the exact nature of the various functional forms arrived at in our investigation. Experimental research has done very much to fill the gap. Yet it will be clear that any attempt to discover the form of a function of a number of variables by means of observation cannot but be very laborious. If the function only involves a single variable, the process is comparatively simple and expeditious. Thus in the case of a body of given shape moving in a direction fixed relatively to the body we found (§ 31) that

$$R = \rho l^2 U^2 \psi\left(\frac{\nu}{Ul}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

ψ contains only the variable ν/Ul, so that a single series of experiments carried out with different values of ν/Ul, *care being taken that the motion is always in the same direction relatively to the body*, will suffice to define ψ graphically, and an empirical formula can be arrived at, if necessary. But where a function of two or more variables has to be discovered, the experimental process is not so simple. Thus in the same case, but with the elasticity of the medium taken into account, where (§ 33)

$$R = \rho l^2 U^2 \chi\left(\frac{\nu}{lU}, \frac{V}{U}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

to determine χ we must observe R for a set of conditions in which V/U has a certain fixed value; then for another set with V/U another fixed value, and so on. Empirical formulæ are then very difficult to construct, and even the graphical representation is not simple. When we come to

the forces and couples and the resistance derivatives in the case of a body moving in any manner, the number of variables is so large that we can never hope to disentangle them by means of experimental observation and graphical representation.

This restriction on our knowledge cannot but be prejudicial to our efforts to discuss in full the dynamics of aeroplanes. In modern aeroplane practice the motions are so complicated, and deviate so much from anything like steady motion, that any accurate information of the forces and couples is at present out of the question, so that the dynamics of, say, looping the loop must remain shrouded in considerable mystery.

But as far as the question of aeroplane stability is concerned, the prospect is more hopeful. Thus consider the case of an aeroplane which is flying steadily in a straight line. Neglecting the elasticity of the air, we have, *e.g.*,

$$R_1 = \rho l^2 U^2 \chi_x\left(\frac{\nu}{lU}, \frac{U_1}{U}, \frac{U_2}{U}, \frac{U_3}{U}, \frac{l\Omega_1}{U}, \frac{l\Omega_2}{U}, \frac{l\Omega_3}{U}\right) \quad . \quad . \quad . \quad . \quad (33)$$

Take the axis of X along the direction of steady motion; then we have

$$U_2 = U_3 = \Omega_1 = \Omega_2 = \Omega_3 = 0,$$

so that

$$R_1 = \rho l^2 U^2 \chi_x\left(\frac{\nu}{lU}, 1, 0, 0, 0, 0, 0\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Thus χ_x is now a function of only one variable, ν/lU, and this can be discovered for any given shape of body. In the same way R_2, R_3, G_1, G_2, G_3 are all obtained by multiplying $\rho l^2 U^2$ into functions of the single variable ν/lU. Further, the resistance derivatives a, b, c, d, e, f are all given by ρl^2 multiplied into functions of the same single variable ν/lU.

The significance of these results cannot be exaggerated. The quantities R_1, R_2, R_3, G_1, G_2, G_3, as well as all the resistance derivatives, can be determined experimentally by means of observations on a model of the aeroplane, care being taken to multiply the result in each case by the dimension factor for the size, the correct velocity factor, and the appropriate function of ν/lU_1.

It is thus clear that the stability of rectilinear steady motion of the aeroplane can be discussed with the help of comparatively simple experiments. We shall see later (Chapter VII.) that useful information is also obtainable with practically no work of an experimental character beyond that required to establish the law of resistance for a plane lamina.

If, however, the steady motion of the machine is not in a straight line, but in a circle or in a helical curve (see Chapter IV.), additional complications are introduced, since in each such case not all the quantities U_2, U_3, Ω_1, Ω_2, Ω_3 are zero. It follows that fuller experimental work is required for the discussion of such motion and its stability.

39. **All Instantaneous Motions Assumed Steady Motions.**—It will be noticed that in all these arguments we have omitted the time element in any explicit form; in other words, we have supposed that the motion of the body and in the air is in each case *steady motion*, so that the effect of the air depends only on the properties of the medium, the shape and size of the body, and the circumstances of the motion. So long as the motion of the body is steady, and this steady motion has been going

on long enough to establish steady motion in the air, this assumption is justifiable. But when the motion is variable, we can easily imagine that this is no longer the case, and, strictly speaking, we should consider that the air forces and couples, as well as the derivatives, should depend explicitly on the time also. It is customary to neglect this consideration. Thus, in varying motion, we find the air effects on the assumption that the instantaneous motion is a steady motion. It is found that this assumption does not introduce any appreciable errors.

CHAPTER II

THE DYNAMICS OF RESISTED MOTION: THE PARTICLE: THE PARACHUTE: THE PHUGOIDS

40. It has been seen that general information on the air resistance on a rigid body moving in air can be obtained from the theory of units. The information thus obtained can be at once used in writing down the equations of motion of such a body. It will be well, however, not only to consider the case of the rigid body of given shape, but also to discuss in outline the chief problems in resisted motion that bear on our subject. The discussion will provide useful knowledge that the student of aeroplane dynamics should possess, and will also help him to grasp the methods and results of the main problem by being led by gradual stages to the final complication that rigid dynamics presents.

We shall commence with the case of a particle.

41. **Particle (Sphere).**—It is the essential nature of a particle that its dimensions are negligible. Thus it is inevitable that the rotations must be neglected and that the shape of the particle must not affect the motion. This is only possible if we assume the particle to be a *very small sphere*, moving with no rotation. We neglect the elasticity of the air. As regards the viscosity, we include its effect by using the formula

$$R = \rho r^2 U^2 \psi\left(\frac{\nu}{rU}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

with the notation of the last chapter, § 29. Experimental and other evidence tends to show that ψ can be put in the form

$$a\ constant + a\ constant\ times\ \frac{\nu}{rU}.$$

Hence for a given sphere and given density and viscosity of the air, we have

$$R = a\ constant\ times\ U^2 + a\ constant\ times\ U.$$

It follows that for a small velocity we can assume

$$R \text{ proportional to } U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and that for a large velocity we can assume

$$R \text{ proportional to } U^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The direction of R will, by symmetry, be through the centre of the sphere along the backward tangent to the path.

The case where R is proportional to U need not detain us here. It

involves sufficiently easy integration if the method to be given for the important practical case is imitated.

42. **$R \propto U^2$.**—To discuss the case of R proportional to U^2, take three rectangular axes, Ox, y, z, as in Fig. 9, and let x, y, z be the coor-

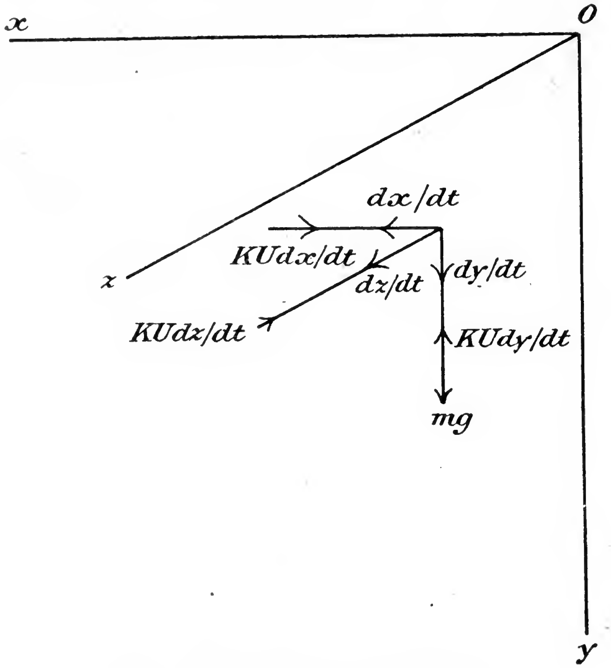

FIG. 9.—Particle in Air: Space Axes.

dinates of the position of the particle at any instant. Let mg be the weight of the particle so that its mass is m. Representing the resistance by

$$R = KU^2,$$

where K is a constant dependent on the size of the particle and the nature of the medium, the components of the resistance are

$$KU^2 \frac{1}{U} \frac{dx}{dt}, \quad KU^2 \frac{1}{U} \frac{dy}{dt}, \quad KU^2 \frac{1}{U} \frac{dz}{dt},$$

i.e.

$$KU \frac{dx}{dt}, \quad KU \frac{dy}{dt}, \quad KU \frac{dz}{dt},$$

along the negative directions of the axes, whilst

$$U^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

In addition to the air resistance, other forces, which we shall call *external forces*, may act on it. Let there be gravity only—this being the important case. We can choose the xy plane to be in the vertical plane in which the particle is projected initially: *the path will lie in this plane*

always, because there will never be any force tending to pull the particle out of it; z is therefore always zero. Let us also choose y to be the downward vertical, and x to be horizontal. Then the external force is mg along y and nothing along x. The equations of motion of the particle are, therefore,

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -KU\frac{dx}{dt} \\ m\frac{d^2y}{dt^2} &= mg - KU\frac{dy}{dt} \end{aligned}\right\}, \qquad (37)$$

in which it is assumed that U is always positive and $U^2=(dx/dt)^2+(dy/dt)^2$, and that dx/dt, dy/dt are both taken with their proper signs. (The student should endeavour to grasp the value of this precaution.)

A few minutes spent in trying to solve the equations (37) will soon convince one that they cannot be integrated with any ease. One special case is suggested, and this is really the most important of all. This is when the motion is always in one vertical straight line, so that $x = 0$. We now have

$$U^2 = \left(\frac{dy}{dt}\right)^2, \text{ so that } U = \left|\frac{dy}{dt}\right|,$$

i.e. we must use for U the numerical value, necessarily positive, of the velocity, whether it is along the positive or along the negative direction of y. Such a method is evidently unsuitable for mathematical analysis (which must perforce consist of transformations and other jugglery). We, therefore, consider the case of upward motion and the case of downward motion separately. Here we shall discuss the case of *fall* under gravity, because it leads on to the theory of the parachute and the theory of gliding flight.

43. **Vertical Fall.**—The equation of motion is

$$m\frac{d^2y}{dt^2} = mg - K\left(\frac{dy}{dt}\right)^2, \qquad (38)$$

since the weight acts downwards, and in a fall the air pressure is a resistance and acts upwards. Writing $U = dy/dt$, so that U is the velocity downwards, the equation of motion is

$$\frac{dU}{dt} = \frac{d^2y}{dt^2} = g\left(1 - \frac{U^2}{k^2}\right),$$

where $k^2 = mg/K$, giving

$$t = \frac{k^2}{g}\int\frac{dU}{k^2 - U^2} - A,$$

where A is an arbitrary constant depending on the initial condition.

Three sub-cases occur. (i) If the initial velocity U_0 is less than k, we write

$$t = \frac{k}{2g}\log_e\frac{k+U}{k-U} - A,$$

so that, as $U = U_0$ at $t = 0$, we get

$$A = \frac{k}{2g}\log_e\frac{k+U_0}{k-U_0},$$

and A is a real positive quantity, since $U_0 < k$. Solving for U in terms of t, we get

$$U = k\tanh\frac{g}{k}(t+A) \qquad (39)$$

As t increases, $\tanh g\,(t + A)/k$ increases, but asymptotically to unity, as shown in Fig. 10, so that U is always less than k, but increases continuously, becoming asymptotic to the value k. Thus if the initial velocity is

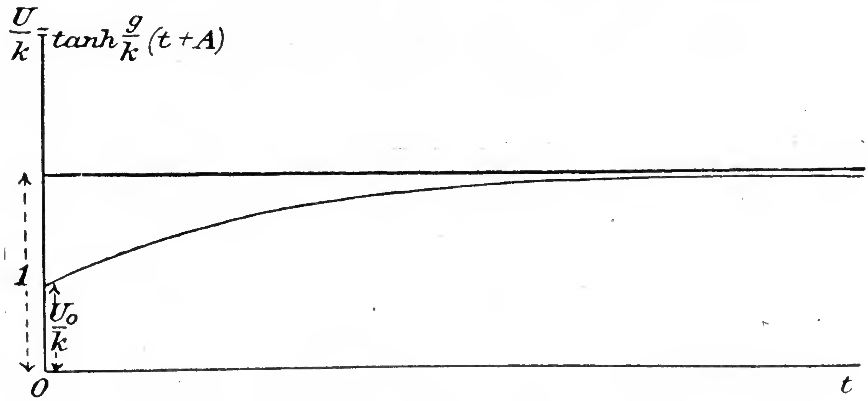

FIG. 10.—Body Falling Vertically: Initial Velocity Less than Terminal Velocity.

less than k, the velocity always remains less than k, which is a limiting value that can never be passed.

To get the velocity in terms of the position, we write (*what the student should remember in all dynamical work*)

$$\frac{d^2y}{dt^2} = \frac{d}{dy}\left\{\tfrac{1}{2}\left(\frac{dy}{dt}\right)^2\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

i.e. *acceleration* = *space differential of kinetic energy per unit mass.* We find

$$\frac{d}{dy}U^2 = 2g\left(1 - \frac{U^2}{k^2}\right),$$

and

$$y = \frac{k^2}{2g}\int\frac{d(U^2)}{k^2 - U^2} + B,$$

where B is another arbitrary constant depending upon the position initially. With the initial velocity less than k, we get

$$y = B - \frac{k^2}{2g}\log_e(k^2 - U^2),$$

and, measuring y from the initial position, so that $y = 0$ at $t = 0$, we have

$$B = \frac{k^2}{2g}\log_e(k^2 - U_0^2),$$

this, like A, being a real quantity. We deduce

$$\left.\begin{aligned} y &= \frac{k^2}{2g}\log_e\frac{k^2 - U_0^2}{k^2 - U^2}, \\ &\text{whence} \\ U^2 &= k^2\left\{1 - \frac{k^2 - U_0^2}{k^2}e^{-\frac{2gy}{k^2}}\right\}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This, too, shows that U is always less than k, gradually approaching it as a limit when y increases indefinitely.

In particular, if the body starts from rest (or, speaking mathematically, if the initial conditions are $y=0$, $U=0$ at $t=0$), we have the simpler results

$$\left.\begin{aligned} U = k\tanh\left(\frac{gt}{k}\right),\quad U^2 = k^2\left\{1-e^{-\frac{2gy}{k^2}}\right\},\\ y=\frac{k^2}{2g}\log_e\frac{k^2}{k^2-U^2}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (42)$$

(ii) If, however, $U_0 > k$, we write

$$t=\frac{k}{2g}\log_e\frac{U+k}{U-k}-A',$$

where

$$A'=\frac{k}{2g}\log_e\frac{U_0+k}{U_0-k}.$$

We find

$$U = k\coth\frac{g}{k}(t+A'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

so that as t increases, U diminishes, but asymptotically to the value k, since (see Fig. 11) $\coth g(t+A)/k$ diminishes asymptotically to unity.

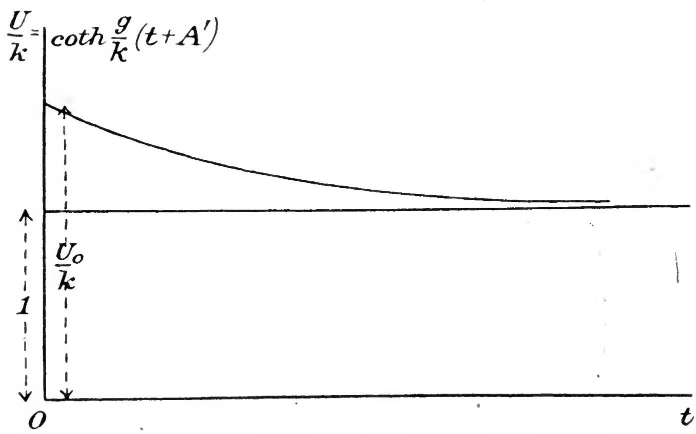

FIG. 11.—Body Falling Vertically: Initial Velocity Greater than Terminal Velocity.

It is also easy to show, in the same way as for $U_0 < k$, that when $U_0 > k$, U is given by

$$U^2 = k^2\left\{1+\frac{U_0^2-k^2}{k^2}e^{-\frac{2gy}{k^2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Thus U is always greater than k and diminishes to k as a limiting value as y increases indefinitely.

(iii) The intermediate sub-case, in which $U_0 = k$, gives

$$\frac{d^2y}{dt^2}=0 \text{ initially},$$

so that the velocity does not begin to change. This, therefore, goes on indefinitely, and we get

$$U = k \text{ always}, \quad y = kt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

y being measured from the position at $t=0$.

44. We therefore deduce that if a spherical particle is projected vertically downwards with initial velocity $k(=\sqrt{mg/K})$ the velocity remains constant, the particle descending uniformly. If the initial velocity is less than k, the velocity increases to k as a limit; and if it is greater than k, it decreases to k as a limit. The three cases are shown diagrammatically in Fig. 12.

The velocity k is, therefore, one that is never passed through; the particle can never begin with a velocity less than k and then acquire one greater than k, or *vice versa*. Also, since for both $U_0 < k$ and $U_0 > k$ we have

$$U^2 - k^2 = (U_0{}^2 - k^2)\,e^{-\frac{2gy}{k^2}}, \qquad (46)$$

and a negative exponential function diminishes to zero quickly, it follows that after a certain time depending on the initial velocity and k, or after

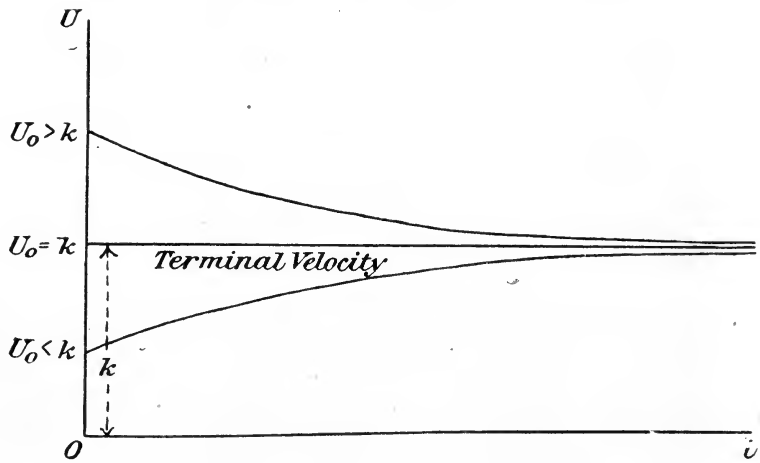

FIG. 12.—Body Falling Vertically: Complete Solution.

a certain fall, the velocity is to all intents and purposes equal to the limiting velocity k.

We call k the **terminal velocity.**

The velocity of a particle falling in air is therefore practically constant after an interval, *e.g.* raindrops have uniform motion when they have fallen a sufficient distance. The importance of this result arises from the fact that it is not only true for a small spherical particle, but also for *any* sphere falling vertically with no rotation (the buoyancy of the air being neglected). It is true not only for a sphere, but also for any body possessing such symmetry about a line that if it falls vertically with this line vertical, the air resistance is along this line upwards.

45. **The Parachute.**—The practical application is to the **parachute.**

Let us consider the quantity k. Its value is $\sqrt{mg/K}$, mg being the weight of a symmetrical body falling symmetrically, and K being the corresponding coefficient of resistance, so that $R = KU^2$. It has been shown that for similarly shaped bodies moving similarly, R is proportional

to the square of a length defining the size. Hence, if we make the area of the parachute large enough we can make K so big that k will become a comparatively small quantity.

The construction of the parachute is as in Fig. 13, there being a more or less convex "umbrella" S, with ropes ending in a means of holding on. If we let S represent the area of the umbrella in square feet, projected on a horizontal plane, experiment gives

$$R \text{ (in lb.)} = 0{\cdot}00125\, SU^2,$$

U being measured in feet per second.

Let us, then, suppose that a parachute of 600 sq. ft. in area and weight 32 lb. carries a passenger of weight 12 st. We get

$$R = \tfrac{3}{4}U^2 \text{ lb.}, \quad mg = 200 \text{ lb.},$$

so that

$$k = \sqrt{200/\tfrac{3}{4}} = 16{\cdot}3 \text{ ft. per sec.},$$

a velocity of descent sufficiently small for safety, representing, in fact, a jump from a height of only a little more than 4 ft.

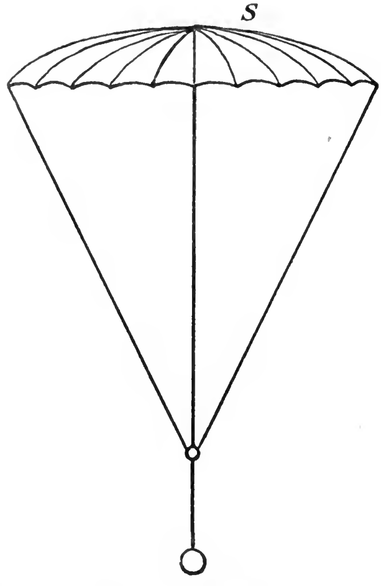

FIG. 13.—Parachute.

One of the chief difficulties in the use of the parachute is the fact that it cannot be carried open and dropped open, but has to be carried closed in the airship, balloon, kite, or aeroplane, and must open as it begins to fall. So long as it is closed it falls practically like a body under gravity with no resistance, so that if it does not open for only a few seconds, the velocity acquired becomes very great. This is dangerous for two reasons: the first is that unless the drop is from a sufficient height the parachute may strike the ground before it opens out; secondly, when it does open out it will require sufficient vertical fall to reduce the velocity to one near the terminal velocity. The best type of parachute is guaranteed to open and "cushion" out in at most 3 secs. Let us adopt the value 3 secs. for the time of free fall. The velocity acquired is 96 ft. per sec., and the distance is 144 ft. The air resistance now comes into action. We have the second sub-case in which $U_0 = 96$, $k = 16{\cdot}3$, giving by (44)

$$\begin{aligned} U^2 &= (16{\cdot}3)^2\{1 + 33{\cdot}6\, e^{-y/4{\cdot}17}\} \\ &= (16{\cdot}3)^2\{1 + (33{\cdot}6)\, 10^{-y/9{\cdot}61}\}, \end{aligned}$$

so that

$$y = 9{\cdot}61 \log_{10}\left\{\frac{33{\cdot}6}{U^2 - (16{\cdot}3)^2}\right\},$$

y being measured from the position where the parachute opens out. Thus even after a further fall of only 9·61 ft. the velocity is reduced to 40 ft. per sec. After another 9·61 ft. the velocity becomes about 19 ft. per sec.,

equivalent to a jump of about $5\frac{1}{2}$ ft. After 28·8 ft. from opening out, U becomes $16\frac{1}{2}$ ft. per sec. It is, therefore, sufficient if the parachute begins to drop from a height of about 173 ft. From a height of anything more than 173 ft. the velocity at the ground is less than $16\frac{1}{2}$ ft. per sec., and the descent is safe.

46. **Spherical Projectile.**—Let us now return to the general case of a particle (or a sphere with no rotation) moving in a vertical plane under gravity and the air resistance. We use the axes defined in § 42. But since the equations of motion there given are not convenient, we shall

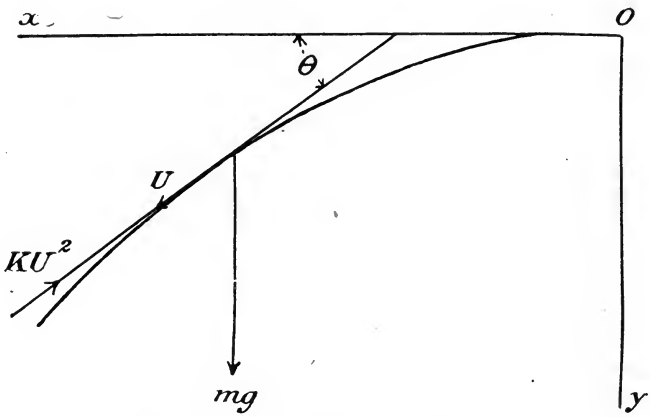

FIG. 14.—Spherical Projectile in Air.

write them in another form, illustrated by Fig. 14. If U is the velocity and θ is the angle its direction makes with the horizontal, the equations of motion along the tangent and normal are

$$\left.\begin{aligned} m\frac{dU}{dt} &= mg\sin\theta - KU^2, \\ m\frac{U^2}{\rho} &= mg\cos\theta, \end{aligned}\right\} \qquad (47)$$

where ρ is the radius of curvature of the path, *i.e.*

$$\rho = \frac{ds}{d\theta}.$$

Using the same definition of k, we get

$$\frac{dU}{dt} = g\left(\sin\theta - \frac{U^2}{k^2}\right),$$

$$\frac{U^2}{\rho} = g\cos\theta.$$

But $dU/dt = d/ds\,.\,(\frac{1}{2}U^2)$, the space differential of the kinetic energy per unit mass,

$$= \frac{d\theta}{ds}\frac{d}{d\theta}(\tfrac{1}{2}U^2) = \frac{1}{\rho}\frac{d}{d\theta}(\tfrac{1}{2}U^2).$$

Hence, substituting in the first equation (47) from the second, we obtain

$$\frac{1}{\rho}\frac{d}{d\theta}(\rho\cos\theta) = 2\sin\theta - \frac{2g\rho\cos\theta}{k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

giving

$$\cos\theta\frac{d}{d\theta}\left(\frac{1}{\rho}\right) + \frac{3\sin\theta}{\rho} = \frac{2g\cos\theta}{k^2},$$

a linear differential equation of the first order. Integrating, we get

$$\frac{\sec^3\theta}{\rho} = A + \frac{2g}{k^2}\int\sec^3\theta\, d\theta,$$

so that

$$\frac{1}{\rho} = A\cos^3\theta + \frac{g}{k^2}\{\sin\theta\,.\,\cos\theta + \cos^3\theta\,.\,\log_e(\sec\theta + \tan\theta)\}, \quad . \quad . \quad (49)$$

where A is an arbitrary constant dependent on the mode of projection. This equation enables us to plot the path. For if we start off with a given direction θ_0 and velocity U_0, the initial radius of curvature is given by

$$\frac{1}{\rho_0} = \frac{g\cos\theta_0}{U_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Putting $\theta = \theta_0$ in the equation (49), we get another value for $1/\rho_0$ in terms of θ_0 and A. Comparing the two values, we get A. The radius of curvature is thus determined at the next point on the path, since the position and direction of a consecutive point are given by taking a small arc on the circle of curvature at the initial point. This process can be continued indefinitely, also backwards, and the path is obtained. The velocity at any point is given by

$$U^2 = g\rho\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Suppose that at the highest point of the path, where $\theta = 0$, the velocity is U_0. Then $\theta_0 = 0$ and $1/\rho_0 = g/U_0^2$; but also $1/\rho_0 = A$ by equation (49), in which θ is put zero. We therefore get

$$\frac{k^2}{g\rho} = \frac{k^2}{U_0^2}\cos^3\theta + \sin\theta\,.\,\cos\theta + \cos^3\theta\,.\log_e(\sec\theta + \tan\theta)\,. \quad . \quad . \quad (52)$$

This can now be used to plot the path both after and before the highest point, it being understood that before the highest point θ is negative, and after the highest point θ is positive. The explicit value of U^2 in terms of θ is

$$\frac{1}{U^2} = \frac{\cos^2\theta}{U_0^2} + \frac{1}{k^2}\{\sin\theta + \cos^2\theta\,.\log_e(\sec\theta + \tan\theta)\} \quad . \quad . \quad . \quad (53)$$

47. **Vertical Asymptote.**—So long as θ lies between 0 and $\pi/2$, every term in the expression for $1/\rho$ is positive. The path is therefore concave downwards everywhere (as is, indeed, obvious from physical considerations). The value of ρ at $\theta = \pi/2$ is ∞.* Thus, ultimately, the path becomes vertical and the body falls with velocity given by putting $\theta = \pi/2$ in the equation (53). We readily find that this velocity is k,* no matter

* It is a well-known result in the calculus that $\underset{x\to 0}{Lt}\, x^a\log_e x = 0$ for $a > 0$.

what the velocity at the highest point may be. The terminal velocity in this general case is thus once more k vertically downwards (Fig. 15).

48. **Inclined Asymptote.**—But let us now investigate the path and velocity before the highest point, *i.e.* while θ is negative. By common sense, U^2 and ρ must both be positive at all points of the path, the former since U must be real, and the latter because the path must be concave downwards everywhere, as the resultant force acting on the body at any

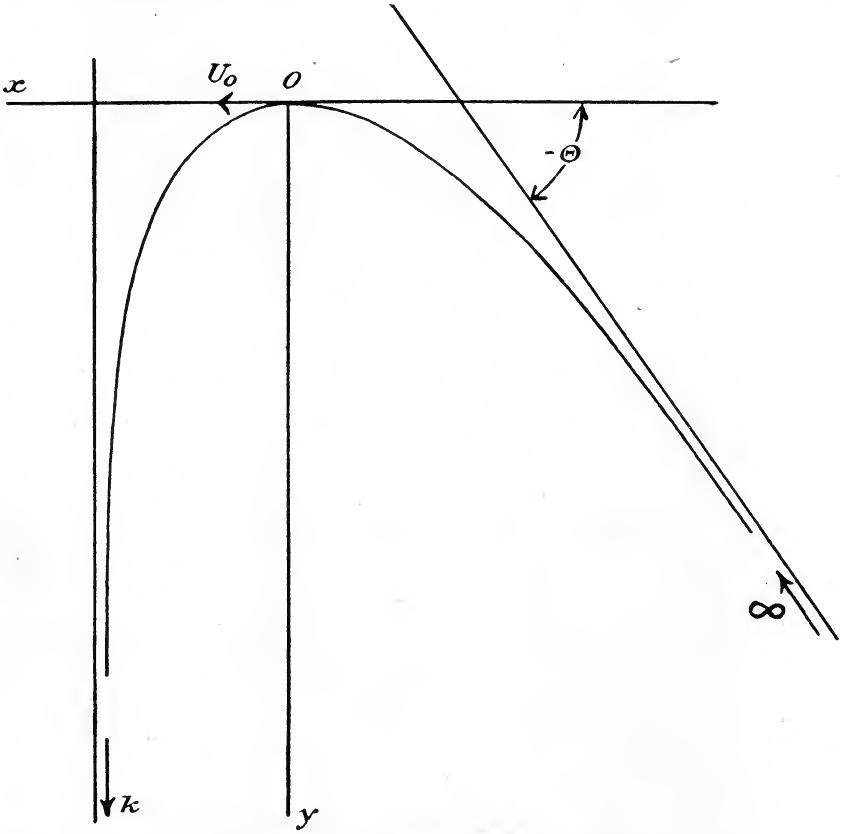

Fig. 15.—Spherical Projectile in Air: Asymptotes.

position always lies below the tangent to the path at the point. For any given value of the velocity U_0 at the highest point, there is a limiting negative value of θ (call it $-\Theta$) at which ρ changes from positive to negative. The angle $-\Theta$ must, therefore, be the limiting value possible for θ; at this angle we get $1/\rho = 0$, and there is, therefore, an asymptote to the path. The arithmetical value of Θ is obtained from the equation

$$\frac{k^2}{U_0^2} = \frac{\sin\Theta}{\cos^2\Theta} - \log_e\left(\frac{1-\sin\Theta}{\cos\Theta}\right)$$

$$= \frac{\sin\Theta}{\cos^2\Theta} + \tfrac{1}{2}\log_e\frac{1+\sin\Theta}{1-\sin\Theta}. \quad \text{. . . . . (54)}$$

We can consider U_0 as a function of Θ or *vice versa*. We find that $d\Theta/dU_0$ is negative, so that the greater U_0 the smaller is Θ, and the smaller U_0 the greater Θ.

Thus the whole path is bounded by two asymptotes, as shown in Fig. 15, one at the infinitely distant vertical end of the path, and the other at the infinitely distant inclined beginning of the path. It is also readily seen that for larger velocity at the highest point the two parts of the path are more widely separated both in angle and in horizontal range at any level (Fig. 16 (*a*)), whilst for smaller velocity at the highest point the asymptotes are closer together and the horizontal range at any level is smaller (Fig. 16 (*b*)). If U_0 is zero, the asymptotes coincide in the vertical up and down path.

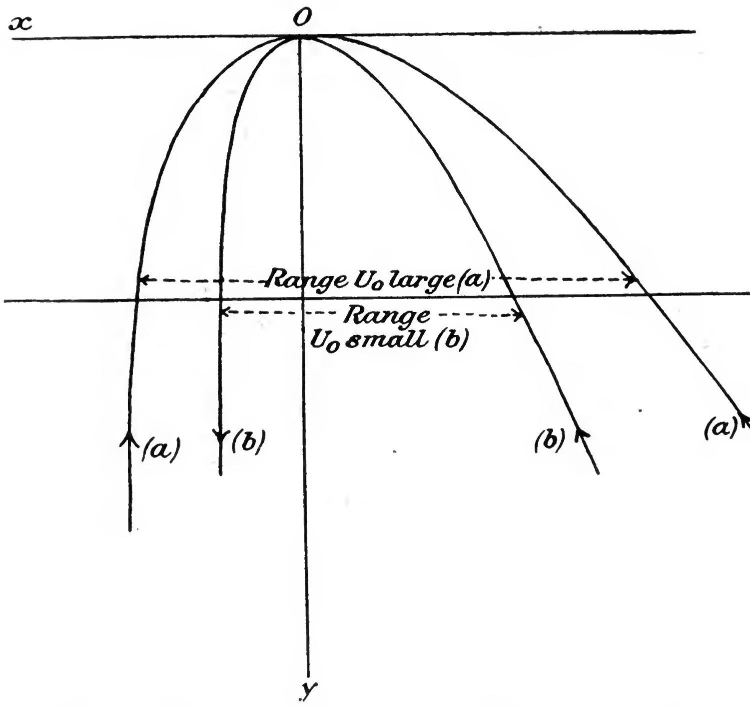

FIG. 16.—Horizontal Ranges for Large and Small Velocity at the Highest Point.

49. **Variation in the Velocity.**—It is interesting to follow out the changes in the velocity. At the infinitely distant beginning of the path, $U = \infty$, since $U^2 = \rho g \cos\theta$, ρ is ∞, and $\cos\theta$ is finite. At the infinitely distant end, $U = k$, as already shown. To discuss the variation in U let us find where the velocity has a maximum or a minimum value. $dU/d\theta = 0$, where $d(U^2)/d\theta = 0$. But $U^2 = g\rho\cos\theta$; hence $dU/d\theta = 0$, where $d/d\theta\,.\,(\rho\cos\theta) = 0$; *i.e.* by equation (48), where

$$\frac{k^2}{g\rho}\sin\theta = \cos\theta,$$

and this is where

$$\frac{k^2}{U_0^2}\cos^3\theta\,.\,\sin\theta + \sin^2\theta\,.\,\cos\theta + \cos^3\theta\,.\,\sin\theta\,.\,\log_e(\sec\theta + \tan\theta) = \cos\theta,$$

which reduces to

$$\cos^3\theta\left\{\sin\theta\left[\frac{k^2}{U_0{}^2}+\log_e(\sec\theta+\tan\theta)\right]-1\right\}=0 \quad . \quad . \quad . \quad . \quad (55)$$

The factor $\cos^3\theta$ vanishes at the vertical asymptote, where $\theta = \pi/2$. Any other stationary points are given by the vanishing of the other factor, *i.e.* where

$$\frac{k^2}{U_0{}^2}+\log_e(\sec\theta+\tan\theta)=\operatorname{cosec}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Now at $\theta = -\Theta$, the left-hand side is equal to $\sin\Theta/\cos^2\Theta$, a positive quantity. Also, as θ increases from $-\Theta$ (which is numerically less than $\pi/2$) to 0, $\log_e(\sec\theta+\tan\theta)$ increases continuously, the graph being concave downwards, and from 0 to $\pi/2$ increases continuously, the graph

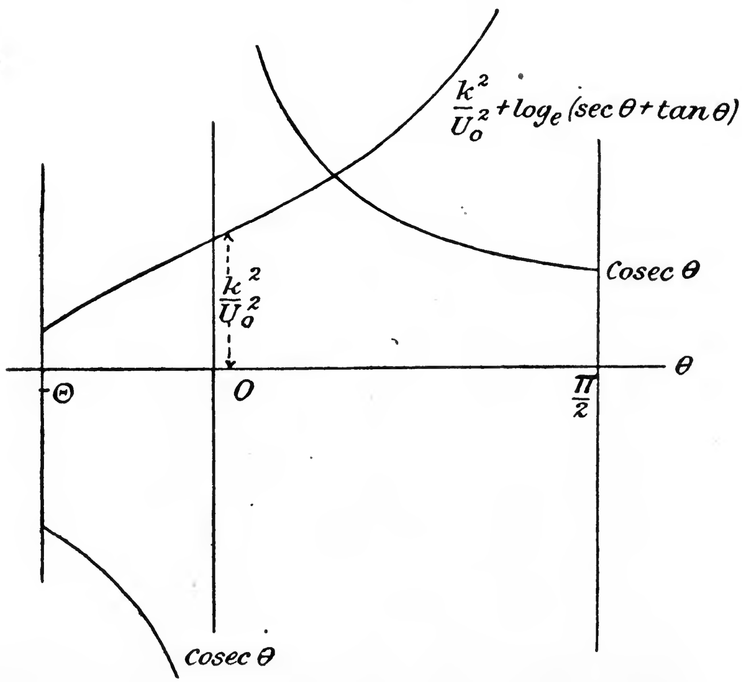

FIG. 17.—Spherical Projectile: Minimum Velocity.

being concave upwards. At the same time $\operatorname{cosec}\theta$ is negative when θ is negative, and when θ is positive it has a graph concave upwards everywhere. The types of functions represented by the left- and the right-hand side of equation (56) are thus as shown in Fig. 17, whence we get that there is only one additional value of θ giving a stationary value of U, and that this value of θ is positive. The value of U must be a minimum at this stationary point, since U decreases from ∞ at the inclined asymptote.

There is, therefore, a minimum value of the velocity at some point *after* the highest point. The figure shows at once that the smaller U_0/k, the smaller is the value of θ for the minimum velocity. If U_0 is zero,

i.e. if the body just rises and falls in a vertical straight line, the minimum velocity is, of course, zero, at the highest point.

50. **Summary.**—We now sum up as follows. The path is contained between an inclined asymptote at the infinitely distant beginning and a vertical asymptote at the infinitely distant end. The velocity starts from infinity at the infinitely distant beginning, decreases as the body rises, keeps on decreasing for a certain time after the highest point is reached, sinks to a minimum, then increases again to the terminal velocity at the infinitely distant end.

The path is determined completely by the ratio U_0/k.

So far k has been assumed given once for all. For different values of k similar results are obtained. It is to be noted, however, that as k increases, *i.e.* as the resistance is less and less intense, the asymptotes recede further and further away on the two sides, and the minimum velocity gets nearer and nearer to the highest point; meanwhile, the path approximates to the parabolic trajectory of elementary dynamics.

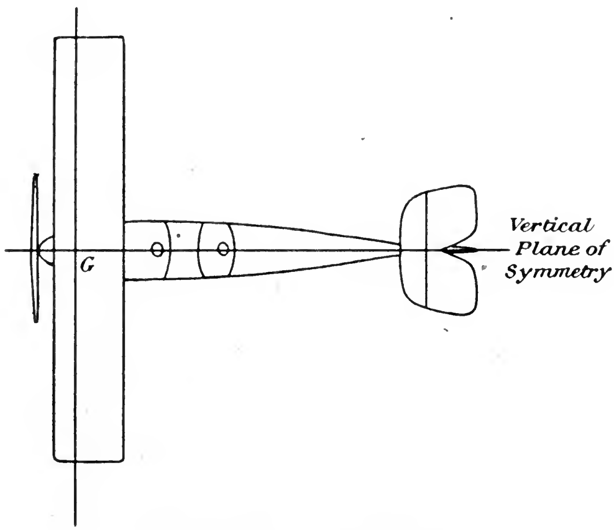

FIG. 18.—Symmetrical Aeroplane.

51. **Longitudinal Motion of a Symmetrical Aeroplane with Small Moment of Inertia.**—Let us now take a body of some size, and to fix our ideas let the shape be that of an aeroplane. It possesses a plane of symmetry perpendicular to the wings. We assume the motion to be in a vertical plane with the plane of symmetry in this plane. Such motion is called **Longitudinal Motion**. If the moment of inertia of the body (Fig. 18) about an axis through its centre of gravity perpendicular to the plane of symmetry is very small, then the moments of the forces acting on the body about this axis must also be small, since otherwise we would get very large angular motion. Let us take the moment of inertia and the moment of the forces to be vanishingly small. Then, if the remaining forces, like gravity and air-screw thrust, pass through the centre of gravity, the resultant air pressure must also pass through the centre of gravity. Now it is obvious *a priori*, and it is verified by experiment, that the resultant

air thrust on a rigid body with no rotation has different lines of action for different directions of the velocity relatively to the body, and the assumption of zero moment of inertia justifies the assumption that the effect of the rotation on the air pressure can be neglected. It follows that the velocity must have a direction fixed in the body; its direction lies, of course, in the plane of symmetry.

Take a point O in the vertical plane of motion and a pair of axes Ox, Oy, as in Fig. 19. At G, the centre of gravity of the body, take a pair of axes GX, GY at right angles to one another and fixed in the body, such that when G is at O and GX along Ox, then GY coincides with Oy. At any moment the velocity U makes a constant angle with GX, both being fixed in the body. Since the rotation is neglected in the air pressure, the effect of the air is a force R, which can be written KU^2, where K is a constant depending on the shape of the body, etc. The direction of R is not along that of U; it is, in fact, the great object of practical

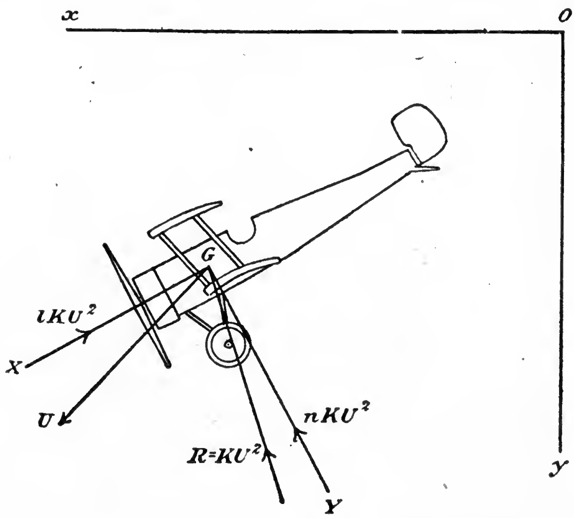

FIG. 19.—Symmetrical Aeroplane: Moment of Inertia Negligible.

aeronautics to make the air pressure as nearly perpendicular to the velocity as possible. We can say *a priori* that the air pressure resultant R will also be fixed in the body, so that the two components along the negative directions of the axes of X, Y are lKU^2, nKU^2 (where l, n are constants connected by the relation $l^2 + n^2 = 1$).

Since the choice of axes GX, GY is at our disposal, so long as we keep them fixed in the body, let us at once choose GX so that it coincides with the direction of the velocity, and GY at right angles. We continue to use lKU^2, nKU^2 for the air-pressure components, it being supposed that l, n have the correct values for the axes thus chosen. Let the curve in Fig. 20 represent the path described by the centre of gravity G, and let θ be the downward angle that the tangent to the path, *i.e.* GX, makes with the horizontal, *i.e.* Ox. If ρ is the radius of curvature of the path measured downwards, and mg the weight of the body, then the tangential

and normal equations of motion are, if gravity is the only external force,

$$\left.\begin{aligned} m\frac{dU}{dt} &= mg\sin\theta - lKU^2, \\ m\frac{U^2}{\rho} &= mg\cos\theta - nKU^2. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (57)$$

The third equation of motion for a rigid body—namely, the equation for the angular acceleration—does not exist, since we assume no moment of inertia and no moment of forces about G.

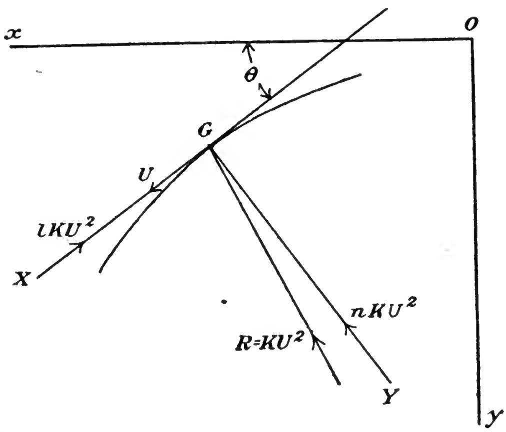

FIG. 20.—Symmetrical Aeroplane: Negligible Moment of Inertia. Tangent and Normal Resolutions.

52. **The Phugoids.**—For an aeroplane, R is so near to GY that we may consider n as practically equal to unity, and l as practically zero. The equations become, Fig. 21.

$$\left.\begin{aligned} m\frac{dU}{dt} &= mg\sin\theta, \\ m\frac{U^2}{\rho} &= mg\cos\theta - KU^2. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (58)$$

These are the equations of motion defining **Lanchester's Phugoids,** and represent one of the earliest and, historically, most interesting attempts at the solution of the motion of an aeroplane. The equations can also be derived by supposing that there is an air-screw along GX giving at any moment a thrust just equal to lKU^2, so that this term disappears in the first equation (57), whilst for nK in the second equation we use a new constant K'. But this type of air-screw thrust cannot be realised in practice, and Lanchester's phugoids must be considered as only an approximation to the solution of aeroplane motion with no air-screw thrust, the approximation being obtained by neglecting the air resistance.

53. **Interpretation in Terms of Dynamics of a Particle.**—It may afford the student a better opportunity of realising the meaning of the phugoids (to be obtained hereafter) if we give an elementary physical interpretation of the problem in terms of the dynamics of a particle, an

interpretation suggested by the fact that the assumption of zero moment of inertia about an axis perpendicular to the plane of motion virtually reduces the body to a particle in so far as motion in this plane is concerned —but, if one may say so, not a spherical particle. The interpretation is as follows.

Imagine a smooth curve in a vertical plane and let a particle be constrained to move under gravity on this curve, the method of constraint being one of the usual devices of theoretical dynamics, as, *e.g.*, taking the particle to be a bead strung on a smooth wire placed along the curve. If R is the normal reaction at any position of the bead, mg its weight, and

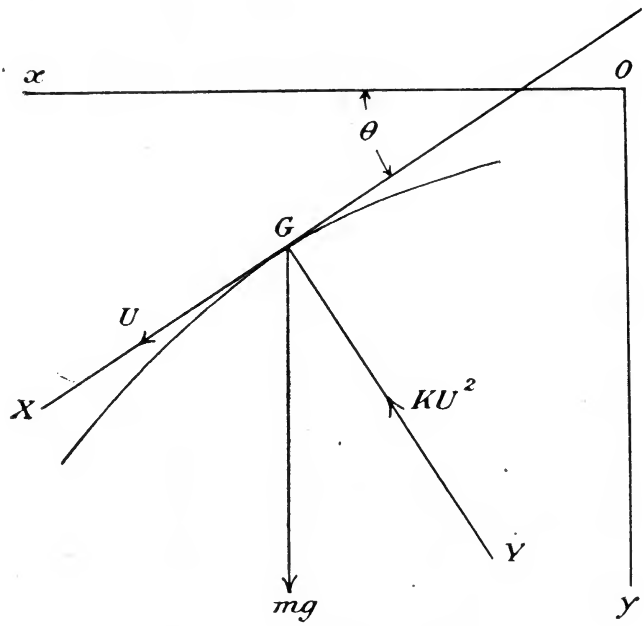

FIG. 21.—Phugoid Problem.

U its velocity, then the equations of motion (Fig. 22) are (using the notation of § 51)

$$m\frac{dU}{dt} = mg\sin\theta, \quad m\frac{U^2}{\rho} = mg\cos\theta - R. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Thus the curve is a phugoid if $R = KU^2$. We have, therefore, the following problem in particle dynamics for the determination of Lanchester's phugoids:

> *To find a smooth curve in a vertical plane such that a particle constrained to move on this curve under gravity experiences a normal reaction proportional to the square of the velocity.*

It is also interesting to note the difference between this problem and the case of the spherical particle of § 46. In the latter we had, using (57), $l = 1$, $n = 0$; here we have $l = 0$, $n = 1$.

54. **Solution of the Phugoid Problem.**—To solve the equations (58), which we write in the form

$$\frac{dU}{dt}=g\sin\theta,\quad \frac{U^2}{\rho}=g\cos\theta-\frac{g}{k^2}U^2, \qquad (60)$$

k being defined by $k^2=mg/K$, we follow Lanchester's method and write

$$U^2=2gy, \qquad (61)$$

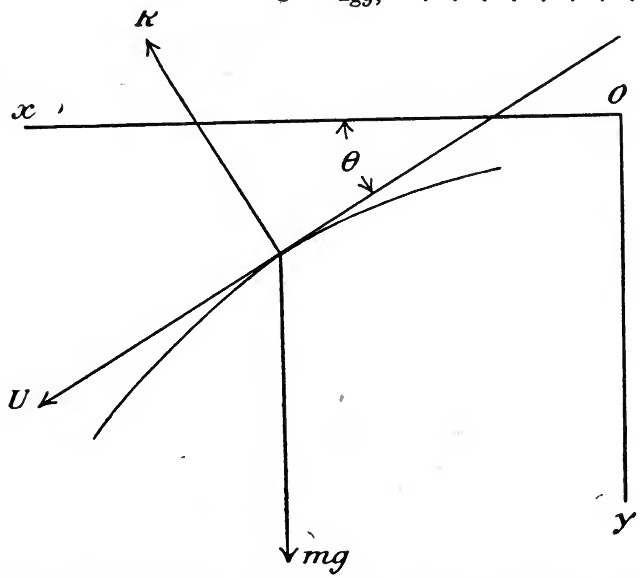

FIG. 22.—Phugoid Problem: Interpretation in Terms of Particle Dynamics.

it being assumed that the depth y is measured from such a level that U is the velocity that would be acquired in falling this distance freely under

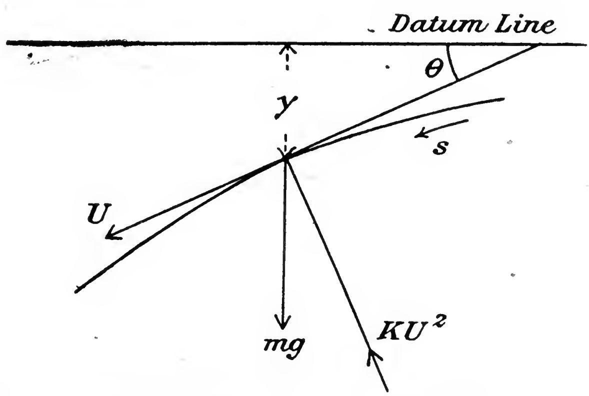

FIG. 23.—Phugoid Problem: Datum Line.

gravity. This level is called the **Datum Line**, Fig. 23. The first equation is now disposed of, since it can, in fact, be written

$$\frac{dU}{dy}\frac{dy}{dt}=g\sin\theta,\quad i.e.\ U\frac{dU}{dy}=g.$$

The second equation becomes

$$\frac{1}{\rho}=\frac{\cos\theta}{2y}-\frac{g}{k^2}.$$

Now

$$\frac{1}{\rho}=\frac{d\theta}{ds}=\sin\theta\,.\,\frac{ds}{dy}\frac{d\theta}{ds}=\sin\theta\frac{d\theta}{dy}=-\frac{d}{dy}(\cos\theta).$$

Hence

$$\frac{d}{dy}(\cos\theta)+\frac{\cos\theta}{2y}=\frac{g}{k^2},$$

a linear differential equation of the first order, which, when integrated, gives

$$\cos\theta=\frac{A}{y^{\frac{1}{2}}}+\frac{2gy}{3k^2}, \qquad (62)$$

where A is an arbitrary constant. The path can now be plotted since we have the inclination at any point in terms of the depth ; or, if we choose, we can follow Lanchester's method and use the radius of curvature, which is given by

$$\frac{1}{\rho}=\frac{A}{2y^{\frac{3}{2}}}-\frac{2g}{3k^2} \qquad (63)$$

The velocity at any point of the path being given by $U^2=2gy$, it follows that if the velocity and direction of projection at any initial point are given, the position of the datum line is determined and A is found from (62).

55. **Special Cases; Semicircular Phugoid.**—The simplest case to consider is $A=0$. This gives $\rho=-3k^2/2g$, *i.e.* a circular path with its centre on the datum line, as is at once made clear from the alternative equation

$$\frac{y}{\cos\theta}=\frac{3k^2}{2g} \qquad (64)$$

Since the velocity becomes zero at the datum line, and above the latter U^2 would be negative, it follows that only the semicircle below the datum line

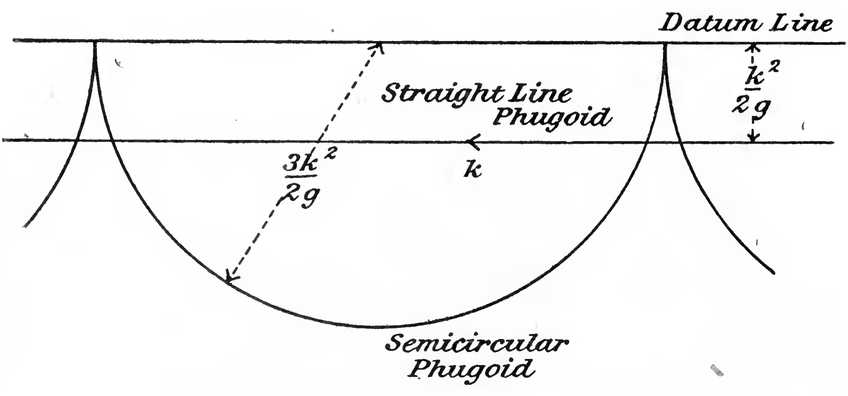

Fig. 24.—Semicircular and Straight-line Phugoids.

can really be described by the body (Fig. 24). When the body reaches the datum line it will either fall back and describe the same semicircle

again, or it may turn suddenly through 180° and describe a fresh semicircle, as in the figure. We need not investigate this point too closely.

56. **Straight-line Phugoid.**—Another simple special case arises for $\rho = \infty$, giving

$$y^{\frac{3}{2}} = \tfrac{3}{4}\frac{Ak^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

so that y is constant. The path is thus a horizontal straight line (Fig. 24) at depth $(\frac{3}{4}\,Ak^2/g)^{\frac{2}{3}}$ below the datum line. Since for this line $\cos\theta = 1$, equation (60) gives $U = k$, and then we get

$$y = \frac{k^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

so that the straight-line phugoid is described with the terminal velocity k at depth $k^2/2g$ below the datum line. It will be seen that the semicircular phugoid has a radius equal to three times this depth.

57. **Given Initial Conditions.**—In general only graphical methods can be used to obtain the phugoids, and the reader is referred to Lanchester's *Aerodonetics* (Chapter III. and Appendices IV., V.) for a full discussion of the method of plotting. We shall here consider the question from the point of view of the initial conditions. It has been said (§ 54) that for given initial conditions the position of the datum line and also A can be determined. This is only partially true, since we get, in fact, two possible values of A.

Let U_0, θ_0 be the initial conditions at some initial point. The equation

$$y_0 = \frac{U_0^2}{2g}$$

determines y_0 with no ambiguity, so that the datum line is uniquely found. But to find A we use the equation

$$A = y_0^{\frac{1}{2}}\left(\cos\theta_0 - \frac{2gy_0}{3k^2}\right) = y_0^{\frac{1}{2}}\left(\cos\theta_0 - \frac{U_0^2}{3k^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (67)$$

in which the factor in brackets is not affected by any ambiguity, whereas the factor $y_0^{\frac{1}{2}}$ can be either positive or negative.

Thus *two phugoids are possible for any given initial conditions*, both with the same datum line. To explain this consider the air pressure.

In Fig. 23 it was tacitly assumed that the pressure is from the under side. But a slight rearrangement of the body can make the air pressure come from the upper side. In the case of a lamina, *e.g.*, if the motion is at a small angle on one side of the lamina, the pressure is in one direction of the normal, whilst if the motion is on the other side, the pressure is in the opposite direction of the normal (Fig. 25). One or the other phugoid is described under given initial conditions. according as one or the other direction of the air pressure is obtained.

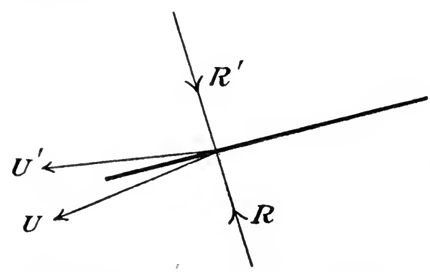

Fig. 25.—Ambiguity in Phugoid Problem.

58. The alternative case (in which A is taken negative) can be discussed analytically by writing the second equation (60) thus:

$$\frac{U^2}{\rho} = g\cos\theta + \frac{g}{k^2}U^2, \qquad (68)$$

whence

$$\left.\begin{aligned} \cos\theta &= \frac{A'}{y^{\frac{1}{2}}} - \frac{2gy}{3k^2} \\ \text{and} \quad \frac{1}{\rho} &= \frac{A'}{2y^{\frac{3}{2}}} + \frac{2g}{3k^2}, \end{aligned}\right\} \qquad (69)$$

A' being an arbitrary constant. The motion is illustrated in Fig. 26.

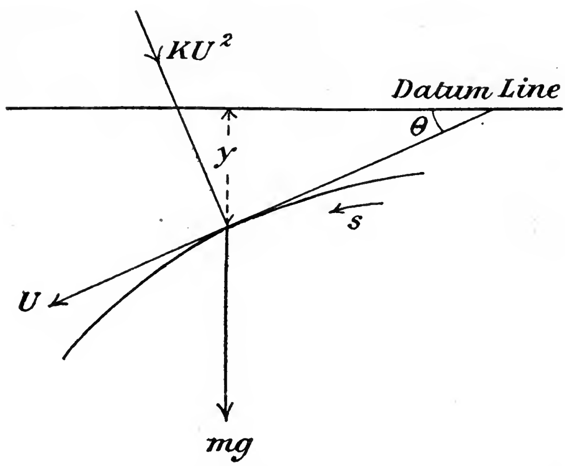

FIG. 26.—Phugoid Problem: Air Pressure from Above.

We now use the positive sign of $y^{\frac{1}{2}}$ everywhere in the set of equations (62, 63) and in the set (69). For each set we get one phugoid for given initial conditions. If these are U_0, θ_0, as before, we have

$$A = \frac{U_0}{\sqrt{2g}}\left(\cos\theta_0 - \frac{U_0^2}{3k^2}\right), \quad A' = \frac{U_0}{\sqrt{2g}}\left(\cos\theta_0 + \frac{U_0^2}{3k^2}\right), \qquad (70)$$

respectively. The radii of curvature at the initial point are

$$\frac{1}{\rho_0} = \frac{g\cos\theta_0}{U_0^2} - \frac{g}{k^2}, \quad \frac{1}{\rho_0'} = \frac{g\cos\theta_0}{U_0^2} + \frac{g}{k^2} \qquad (71)$$

59. **Inflected and Tumbler Phugoids; Catastrophic Instability.**—As an example suppose θ_0 is zero, and $U_0 < k$. Then ρ_0 and ρ_0' are both positive, but $\rho_0 > \rho_0'$. Thus, Fig. 27 (a), if the body is moving horizontally with such a velocity that it is between the corresponding datum line and straight-line path, we get a less curved path and a more curved path. In the former the air pressure is from below, and so long as it is from below we get the path given by the set of equations (62, 63),

shown by Lanchester to be a series of undulations continued indefinitely (before as well as after) and said to be of the **Inflected** type. But in the more curved path the air pressure is from above, and so long as it is from above we get the path given by equations (69), this path being now a series of loops, called by Lanchester a **Tumbler** curve.

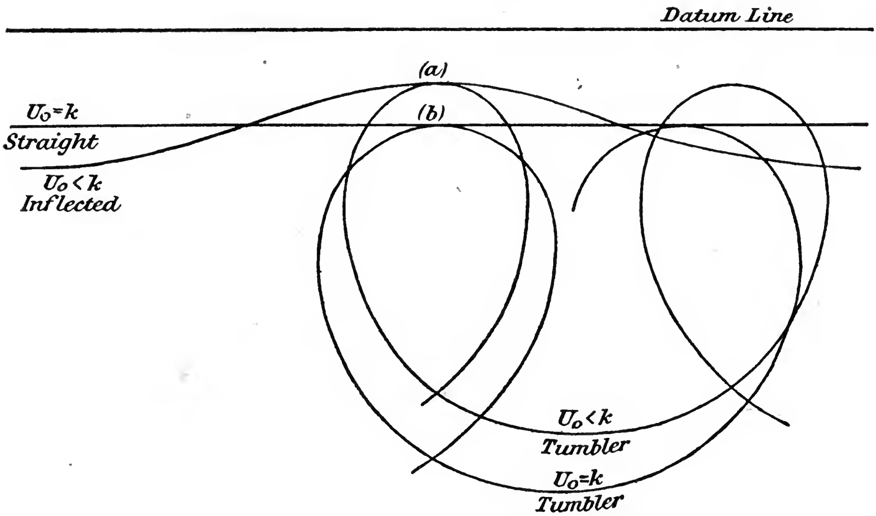

FIG. 27.—Inflected and Tumbler Phugoids.

Now *at any moment* a sudden change from air pressure below to air pressure above can take place, due to slight, and perhaps unrecognisable, causes, as, *e.g.*, a gust of wind. Thus at any moment the path can change suddenly from the inflected path it is describing to that tumbler type of path, which corresponds to the position and direction at the point (Fig. 28).

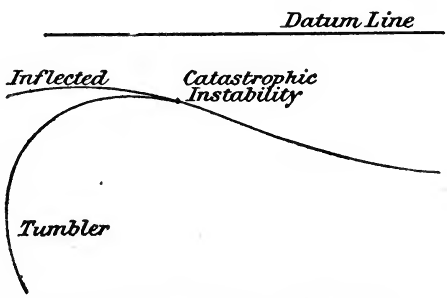

FIG. 28.—Catastrophic Instability.

This is a type of *instability* that can lead to serious consequences, and it has been called **Catastrophic Instability.** It is true that the converse might also happen, but as, if it does happen, it cannot do any harm, we need not consider this possibility any more in detail.

Catastrophic instability can also happen, of course, if U_0 is just equal

to k, *i.e.* for the straight-line path, Fig. 27 (b). When it takes place we have for the tumbler curve

$$A' = \tfrac{4}{3}\frac{k}{\sqrt{2g}}, \quad \frac{1}{\rho_0'} = \frac{2g}{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

If now we come to $\theta_0 = 0$, $U_0 > k$, we have ρ_0 negative, ρ_0' positive. Thus at the initial point the flatter path is concave upwards, and the more

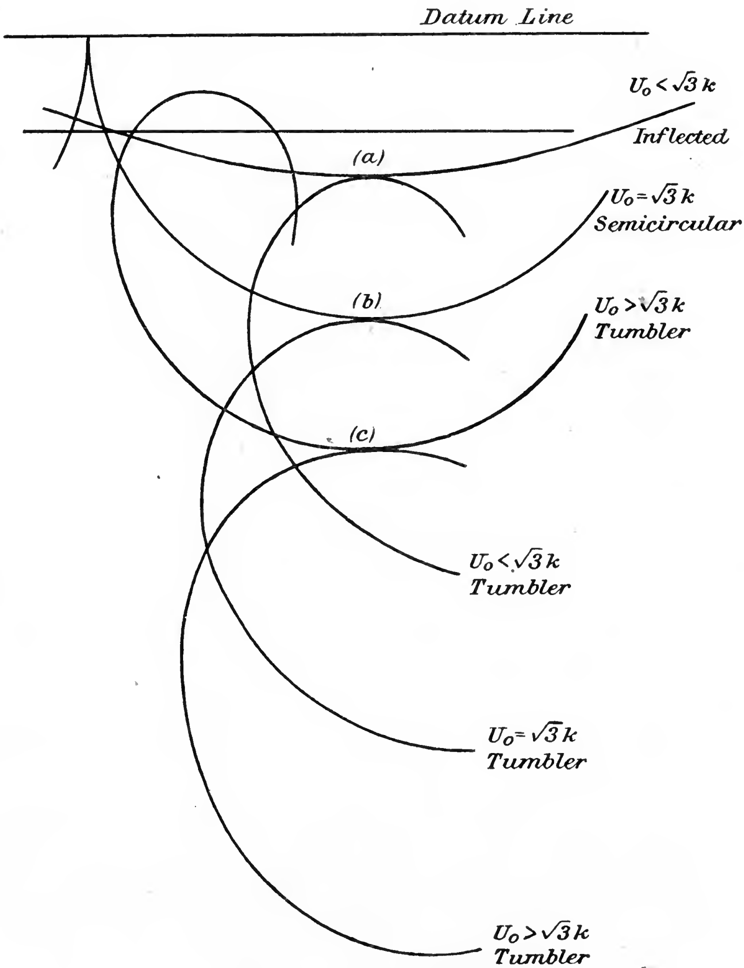

FIG. 29.—Phugoids under Various Conditions.

curved path is concave downwards. It can be shown that, so long as $U_0 < \sqrt{3}k$, *i.e.* the initial point is above the lowest point of the corresponding semicircular path, the flatter path is of the inflected type, and the other is of the tumbler type. When $U_0 = \sqrt{3}k$, the former is the semicircular path, the latter a tumbler. When $U_0 > \sqrt{3}k$, both curves are tumblers. These three cases are all illustrated in Fig. 29 (a), (b), (c).

Fig. 30 (taken from Lanchester's book) shows several typical phugoids drawn carefully from the equations. No. 1 is the straight-line path, No. 7

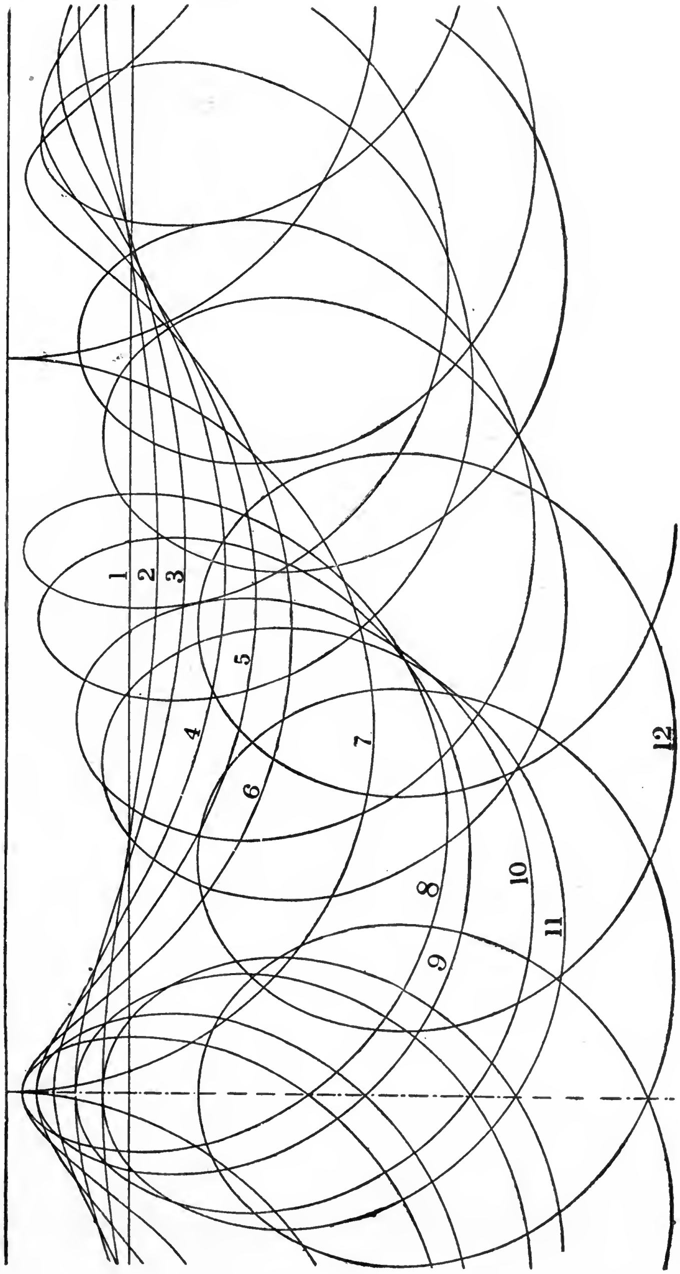

Fig. 30.—The Phugoid Chart. The Flight Path Plotted from the Equation.

is the semicircular path; the top line is the datum line. No. 2 and No. 11, No. 3 and No. 10, No. 5 and No. 9, No. 6 and No. 8, are

pairs of phugoids, each pair for given horizontal velocity. No. 12 is a tumbler path corresponding to an inflected path between the levels of 3 and 4. The case of two corresponding tumblers is not illustrated in this chart.

60. **Scale.**—Nothing has yet been said about the scale of the phugoids. Equations (62), (69) show that if y is changed in the ratio L, while A, U and k are changed in the ratio $L^{\frac{1}{2}}$, then $\cos\theta$ is unaffected and ρ is changed in the ratio L. This means that a chart of phugoids once drawn will suit all possible cases, provided that if the scale of the diagram is changed in any ratio, the scale of the velocity is changed in the square root of this ratio.

61. **Corrected Phugoids.**—Returning to the general problem the solution of which is represented by the phugoids, we must remark that although the problem affords us good information concerning aeroplane

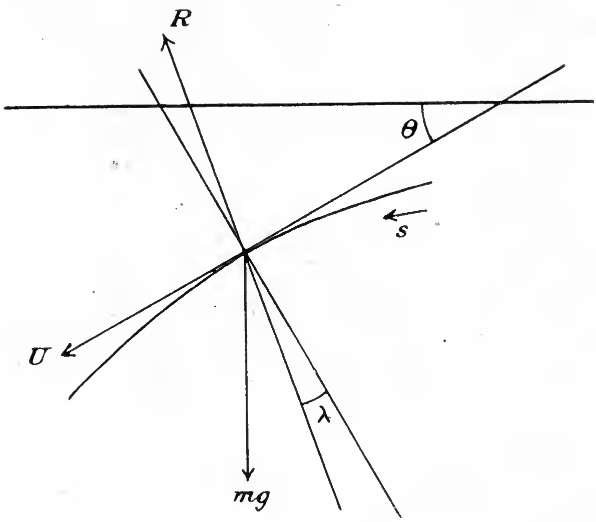

Fig. 31.—Phugoid Problem with Head Resistance Included.

paths, this information is yet not applicable in detail to actual facts. It is true that by the consideration of the paths of the inflected type Lanchester was able to arrive at a condition of stability for an aeroplane flying horizontally. We must, nevertheless, recognise that the neglecting of the head resistance means leaving out a very essential feature of aeroplane flight. The way in which to include the air resistance is at once indicated by the particle dynamics interpretation of the phugoids, § 53.

In this interpretation let us now suppose the curve to be rough, the roughness being represented by an angle of friction λ. As the friction is opposed to the motion, the reaction R of the curve will now be at an angle λ on the same side of the downward normal as the velocity, as shown in Fig. 31. If, then, we put $l = \sin\lambda$, $n = \cos\lambda$, we get for the equations of motion the same equations as (57), if $R = KU^2$. Thus the phugoids corrected for head resistance are given by the solution of the following problem in particle dynamics:

To find a rough curve in a vertical plane such that a particle constrained to move on this curve under gravity experiences a reaction proportional to the square of the velocity, the angle of friction being given.

The equations of motion of the corrected phugoids are (57), which can be written

$$\frac{dU}{dt} = g\sin\theta - gl\frac{U^2}{k^2}, \quad \frac{U^2}{\rho} = g\cos\theta - gn\frac{U^2}{k^2}, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

assuming the air pressure to be from below in the figure as drawn (Fig. 31). For the alternative case of the air pressure being above, we use $+n$ instead of $-n$.

We can no longer use the datum line method in order to obtain U. We must eliminate ρ, and this is done by writing

$$\frac{dU}{dt} = \frac{d}{ds}(\tfrac{1}{2}U^2) = \frac{1}{\rho}\frac{d}{d\theta}(\tfrac{1}{2}U^2).$$

On dividing the equations (73) into one another, we get

$$\frac{\frac{dU}{d\theta}}{U} = \frac{\frac{d}{d\theta}(\frac{1}{2}U^2)}{U^2} = \frac{\sin\theta - \frac{l}{k^2}U^2}{\cos\theta - \frac{n}{k^2}U^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

with $+n$ instead of $-n$ for the alternative case of air pressure above. This is a differential equation for U^2 in terms of θ. The special cases $l=1$, $n=0$; $l=0$, $n=1$ have been integrated as far as a first integral. When both l and n exist the equation seems to be too difficult to integrate at all.

The student must reconcile himself to the occurrence of such difficulties. In formal dynamics as generally taught, it is usual to present for discussion only such problems as are soluble in exact terms by means of some mathematical function, at least as far as a first integration. But in practice we cannot choose our problems in this way. Some means must be found for solving a problem presented by Nature, even though no formal mathematical functions are as yet available for exact solution. This is often the case in aeronautics, as we shall have occasion to discover in the course of this book.

62. **Method of Successive Approximations.**—If we are keenly desirous of discovering the forms of the corrected phugoids, we must make use of the fact that in aeroplane motion the quantity l is as a rule quite small, rarely more than $\frac{1}{10}$. We can, therefore, attempt a solution by means of *successive approximation.*

This is a very important method based on the fact that a function of a small quantity can, as a first approximation, be represented as a known quantity *plus* a small quantity; in symbols

$$f(x) = f(0) + xf'(0),$$

where x is a small quantity, f is a functional form, $f(0)$ is the value obtained by putting $x=0$ in $f(x)$, $f'(0)$ is the value obtained by putting $x=0$ in the function $f'(x)$ derived by differentiating $f(x)$ with respect to x.

In our problem any quantity in the solution, say $\cos\theta$ in terms of U, must involve the quantity l. If, then, l is small, we have

$$\cos\theta = (\cos\theta)_{l=0} + l \times \text{a quantity in which we put } l = 0.$$

The part $(\cos\theta)_{l=0}$ is Lanchester's solution. Substitute this value in the term which contains l in the equation for finding $\cos\theta$, and we get a second approximation. This process can be repeated as often as is thought useful.

The differential equation for U in terms of θ can be put in the form

$$\cos\theta - U\sin\theta\frac{d\theta}{dU} = \frac{n}{k^2}U^2 - \frac{l}{k^2}U^3\frac{d\theta}{dU}.$$

If we include terms involving the first power of l, but neglect higher powers, we can put $n = 1$, since $n = \sqrt{1 - l^2}$. We get

$$\frac{d}{dU}(U\cos\theta) = \frac{U^2}{k^2} - \frac{l}{k^2}U^3\frac{d\theta}{dU},$$

so that

$$\cos\theta = \frac{A}{U} + \frac{U^2}{3k^2} - \frac{l}{k^2 U}\int U^3 d\theta, \quad \ldots \ldots \quad (75)$$

where A is an arbitrary constant, and in the integral we suppose U^3 to be replaced by its value in terms of θ. Neglecting l, we get Lanchester's solution. To include the first power of l we now use in the integral the value of U^3 given by Lanchester's solution.

63. **Glide.**—For practical purposes the straight-line phugoid and the nearly straight inflected paths are the important ones to consider. For the former we get the corrected path at once. A straight line inclined

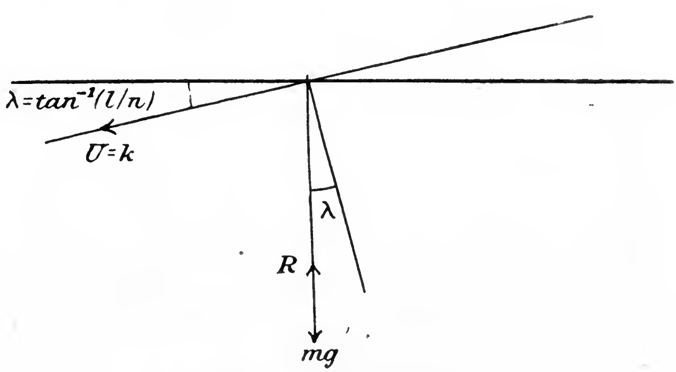

FIG. 32.—Gliding Path.

to the horizontal at the angle $\lambda = \tan^{-1}(l/n)$, Fig. 32, is obviously one that gives constant velocity, since the head resistance just balances the component of the weight down this line; also the normal component of air pressure, nKU^2, can be made equal to the component of gravity down this normal by choosing U so that $nKU^2 = mg\cos\lambda$, which, by definition $k^2 = mg/K$, gives

$$U = k \quad \ldots \ldots \quad (76)$$

This straight-line path is of great importance. It exists for all values of l and n and is called the **gliding path.** Given the correct velocity and direction, any body can glide down a straight line under gravity without any change in the velocity. The importance of this result in practice is due to the fact that we have here a sort of parachute motion, in which, however, the terminal velocity is the velocity all through (§ 43 (iii)). An aeroplane with no propeller, or with zero propeller thrust, can thus glide down in safety, provided the angle $\tan^{-1}(l/n)$ is not too great. This angle, called the **gliding angle,** is one of the most important data of any machine, and the smaller it is, the preferable and the safer the machine.

There is only one gliding angle, because this angle is determined by choosing the direction of motion in such a way that the air pressure passes through the centre of gravity of the machine.

We shall return to this later.

EXERCISES (Chapter II.)

1. Discuss the vertical fall of a body in air, with the resistance varying as the first power of the velocity. If $R = KU$, prove that the terminal velocity is $k = mg/K$, where m is the mass. Shew that

$$U = k\left(1 - e^{-\frac{g}{k}t}\right), \quad y = kt - \frac{k^2}{g}\left(1 - e^{-\frac{g}{k}t}\right)$$

U and y being zero at time $t = 0$.

2. A parachute of area 900 sq. ft. is used with total weight 210 lb. Find the terminal velocity. If it opens out in 4 seconds, find the necessary free fall in order that the velocity may be reduced to 10 ft. per sec.

3. The weight of a parachute with the passenger is 180 lb. Find the necessary area so that the shock on landing may be equivalent to that of a fall from a height of 5 ft. in vacuo.

4. The area of a parachute is 800 sq. ft. Find the maximum total weight so that the landing shock shall not be greater than that in a fall of $4\frac{1}{2}$ ft. in vacuo.

5. If the density of the air increases downwards according to the law $\rho = \rho_0(1 + ay)$ where a is a very small number, find the law of fall of a parachute, taking the air resistance to vary as ρU^2.

6. Do question 5 with the air resistance proportional to ρU.

7. Discuss the motion of a spherical projectile in air with resistance varying as the velocity (assume constant density). Prove that if U_0 is the velocity at the highest point and k is the terminal velocity, then the vertical asymptote to the path is U_0k/g from the highest point, and the minimum velocity is $U_0k/\sqrt{U_0^2 + k^2}$ at time $\frac{k}{g}\log_e\frac{U_0^2 + k^2}{k^2}$ after passing the highest point. Discuss the infinitely distant beginning of the path. (Use Cartesian co-ordinates.)

8. The engines of a ship are stopped whilst the ship is still moving. Find the subsequent motion, taking the water resistance to be proportional to the square of the velocity.

9. Investigate the rectilinear motion of a body acted on by a constant frictional force and an air resistance varying as the square of the velocity. Shew that it is really the friction that makes the body actually come to rest.

10. Write down the equations of motion of a spherical projectile moving in air of variable density, so that $R \propto \rho U^2$ and $\rho = \rho_0 f(y)$: deduce an equation between θ and y.

11. If the body in § 51 is made to move in its plane of symmetry with this

plane horizontal, prove that the path is a circle, with diminishing velocity, if $R \propto U^2$. (See § 61.)

12. Find the path in question 11 with $R \propto U$, and more generally for $R \propto U^n$. Distinguish between n greater than and n less than 2.

13. Prove that the corrected nearly straight phugoids can be represented by the equation

$$\cos(\theta + \lambda) + l\theta = \frac{A}{U} + \frac{U^2}{3k^2},$$

where θ is measured from the direction of glide and λ is the angle of glide. Discuss the path.

14. Examine the effect of variable air density on the shapes of the phugoids.

CHAPTER III

LONGITUDINAL MOTION OF THE AEROPLANE (RIGID BODY): THE PARACHUTE: THE KITE

64. No matter how we correct Lanchester's solution of the aeroplane problem, we shall still be far from the truth because of the assumptions made, viz. zero moment of inertia and zero couple due to air pressure. We must attack the more general problem in which both assumptions are avoided. The effect will be twofold: the air pressure will cause changes in the angular motion, and these will in turn affect the air pressure.

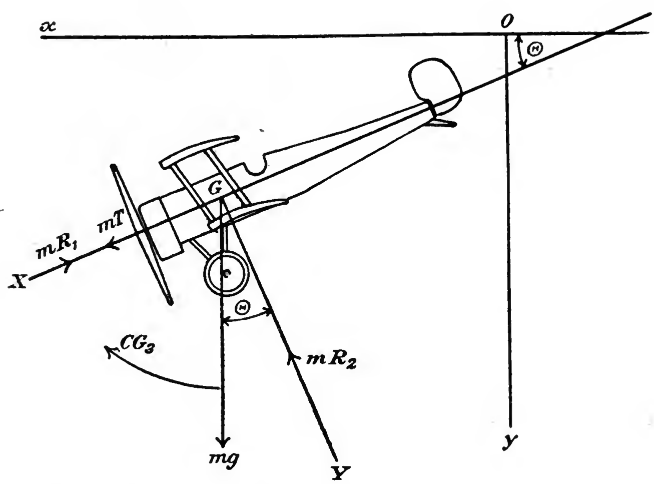

FIG. 33.—Longitudinal Motion of the Aeroplane: Space Axes and Body Axes.

We use the axes in § 42, Fig. 9. We assume symmetry in the machine and in the motion; we then have only two co-ordinates, x, y, the former horizontal to the left, the latter vertically downwards. At G, the centre of gravity of the machine, take two axes GX, GY fixed in the machine, in such a way that when G is at O and GX along Ox, then GY is along Oy (Fig. 33). Let x, y be the co-ordinates of G, and let Θ be the angle that GX makes with Ox, measured from x to y. Whatever the motion, there will be, due to the air pressure, forces R_1, R_2 per unit mass of the machine along the negative directions of the axes GX, GY, and

a moment CG_3 reckoned positive from Y to X, C being the moment of inertia about G in the plane of motion. The equations of motion are two equations for the motion of the centre of gravity as if the machine were a particle at G acted on by all the forces, and a third equation for the rotation of the rigid body. The last equation can be written down in only one way, but for the first two we have to decide on the most convenient forms. Shall we use the fixed directions Ox, Oy for decomposing the accelerations and forces? Or shall we use the tangent and normal to the path? Or shall we use some other more convenient way?

65. **Motion Referred to Fixed Axes.**—Let us try the first method. We must put down symbols for any external forces that may act. If m is the mass, then mg is the weight; we need not consider anything in addition except the propeller thrust. Since the propeller axis is fixed in the body, we simplify matters by choosing GX parallel to this axis. *Assuming T to pass through G*, we put the thrust T per unit mass, as shown in the figure; we neglect any gyroscopic effects due to the rotation of the propeller and engine if of rotary type. The equations of motion are:

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= R_2\sin\Theta - R_1\cos\Theta + T\cos\Theta,\\ \frac{d^2y}{dt^2} &= g - R_1\sin\Theta - R_2\cos\Theta + T\sin\Theta,\\ \frac{d^2\Theta}{dt^2} &= -G_3.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

The important special case of steady motion, *i.e.* constant velocity in a straight line, can be discussed by means of these equations.

66. **Longitudinal Steady Motion.**—For steady flight we have no accelerations and Θ is constant. The conditions are, therefore,

$$\left.\begin{aligned}R_2\sin\Theta - R_1\cos\Theta + T\cos\Theta &= 0,\\ g - R_1\sin\Theta - R_2\cos\Theta + T\sin\Theta &= 0,\\ G_3 &= 0.\\ \text{The first two equations give}\qquad&\\ T = R_1 - g\sin\Theta,&\\ R_2 = g\cos\Theta.&\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The condition $G_3 = 0$, *i.e.* that there shall be no moment due to the air pressure, implies that the pressure must pass through the centre of gravity, which is a fixed point in the machine. We, therefore, expect *a priori* that *for any one form of machine there is one direction of steady motion relatively to the body.* This, therefore, makes R_1, R_2 constants for given velocity, so that Θ and T must be constant. Thus steady flight is only possible with constant propeller thrust. The direction of flight is also fixed in space, since it is fixed in the body and the body has a constant orientation Θ.

Conversely, with a given machine, fixed in shape, and with a given thrust, since the direction of motion relatively to the machine is given, the ratio R_1/R_2 is determined. Hence by (78) Θ is given, and therefore the velocity is also given, as R_1, R_2 depend on the velocity. Thus there are a fixed velocity, a fixed direction of steady flight, and a fixed orientation of the body. To vary the velocity it is not sufficient to vary the thrust, for this will also cause a variation in the direction of flight.

67. **Graphical Statics; Elevator.**—To illustrate this very important point let us consider the case of horizontal flight, with the propeller axis also horizontal, Fig. 34 (*a*). Let mR, the resultant air resistance, have the direction shown, and draw the triangle of forces. We get, Fig. 34 (*b*), a right-angled triangle rtw. Now suppose that the body has the same shape, but that the velocity is made different in amount, yet in the same direction relatively to the body. Then mR is changed to, say,

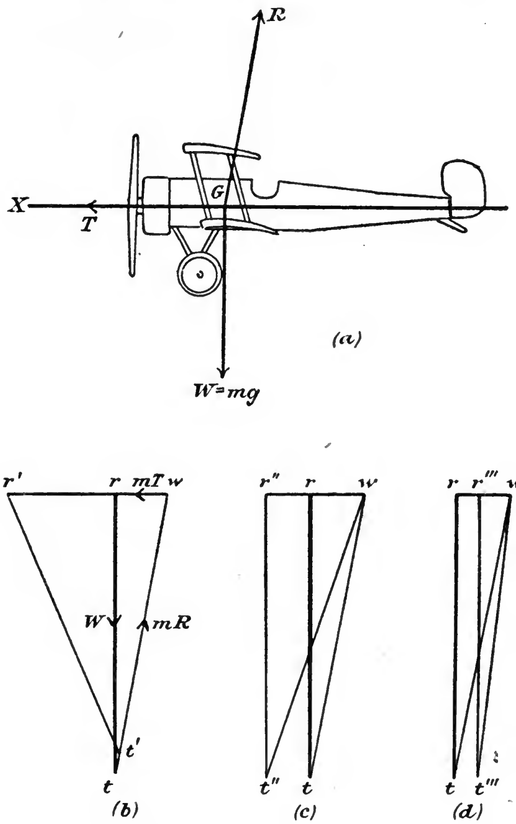

FIG. 34.—Longitudinal Steady Motion of the Aeroplane. Graphical Statics.

mR', keeping its direction relatively to the body the same; mT is now mT', also fixed in the body. But W must remain the same. Take then

$$wt' = \frac{R'}{R} wt, \quad wr' = \frac{T'}{T} wr;$$

we must have $t'r' = tr$, since both represent W on the same scale, and clearly $t'r'$ is not in the same direction as tr, *i.e.* the vertical direction is changed relatively to the machine; in other words, the machine flies at a different inclination.

If the machine is to continue flying with the general position of the body unaltered, *i.e.* with the propeller axis still horizontal, we

must, as shown in Fig. 34 (*c*), get wt'' for the new air pressure and wr'' for the new thrust in such a way that $t''r''$ is equal and parallel to tr, which means that the *direction* of the air pressure must be changed at the same time as its amount is changed. There must then be some change in the shape.

This is provided by means of the elevator, which in modern machines is at the back, being, in fact, part of the "tail" (Fig. 35). By means of control wires, the pilot is able to turn the elevator either up or down, at the same time as he makes adjustment in the throttle of the engine so as to regulate the thrust. By turning the elevator up and increasing the thrust correspondingly, he gets the arrangement in Fig. 34 (*c*), which shows the case of decreased velocity, but descending. By turning the elevator down and diminishing the thrust correspondingly, he gets Fig. 34 (*d*), which shows the case of increased velocity, but ascending.

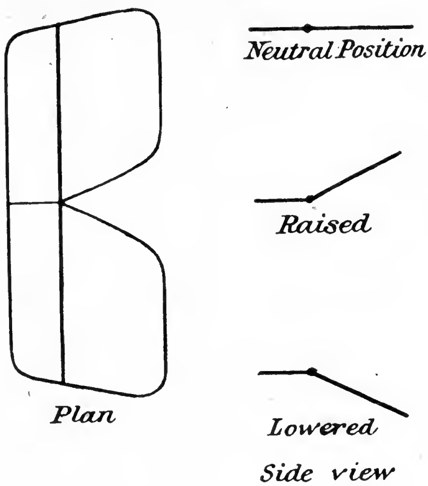

FIG. 35.—Elevator.

68. **General Theory of Steady Motion.**—To discuss the general problem of steady motion graphically, let mR, the resultant air resistance, make an angle θ with the Y axis, as in Fig 36 (*a*). We again take the triangle of forces twr, Fig. 36 (*b*). To change the velocity, keeping the orientation of the body fixed, we must get a triangle like $tw'r$, or $tw''r$, in which w', w'' are on the line wr. It follows that the direction of the resultant pressure must also change so as to be parallel to tw', tw'' respectively; in other words, the elevator must be manipulated and the engine modified accordingly. The direction of motion in space is changed.

If, on the one hand, the propeller thrust is changed and no corresponding elevator change is introduced, we must have a different direction of the body; the velocity direction must be fixed in the body to satisfy the condition $G_3 = 0$, and so the air-pressure direction is fixed in the body. There is thus a constant angle between the propeller thrust and the air pressure. If, then, we draw the circumcircle through twr, the new thrust rw' (*i.e.* $> rw$) being given, we get w' and, therefore, the direction of rw', *i.e.*

in the case shown in Fig. 36 the new direction of the propeller axis is at a smaller angle with the horizon, Fig. 36 (*c*). If the new thrust is rw'' (*i.e.* $< rw$), we get the new direction of the propeller axis at a greater angle with the horizon. The motion is along the same direction as seen from the machine.

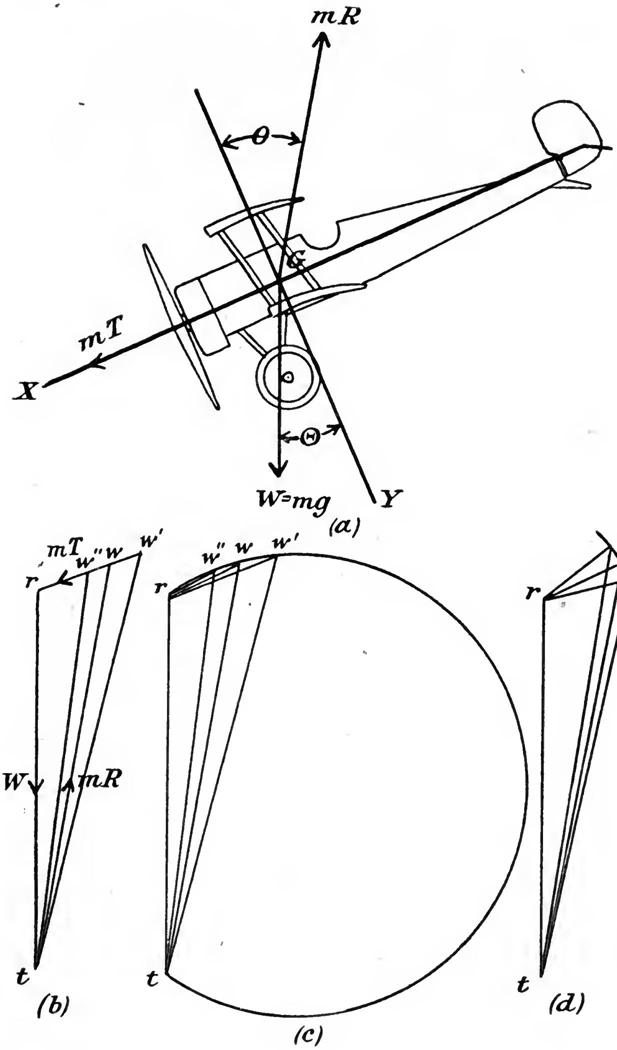

FIG. 36.—Graphical Statics of Steady Motion (*a*) of the Aeroplane; (*b*) Direction of Velocity and Orientation of Body Unchanged; (*c*) Altered Thrust, no Change in Elevator; (*d*) Unchanged Thrust, Elevator Turned.

If, on the other hand, the thrust is kept constant and the elevator is used, the velocity direction is in some different position in the body, determined by the amount of change in the elevator; also the angle between the thrust and the air pressure is changed. Both these changes are presumably given by the given elevator turn. In the triangle of forces

twr, with r as centre and rw as radius, draw a circle. The new direction in space of the thrust is a radius rw' of this circle. Also the angle $rw't$ is the given new angle between the thrust and the air pressure; describe the arc $rw't$ on rt containing this angle. The intersection gives the position of w', *i.e.* the new direction of the propeller axis. It is seen that in the case in Fig. 36, if the acute angle between the thrust and air pressure is increased (rw'), the direction of the body in steady flight is made more horizontal, and if decreased (rw''), then the direction of the body is lowered, Fig. 36 (*d*). The former occurs when the elevator is turned down, the latter when it is turned up.

In general, when both the elevator is turned and the thrust changed, we get some different case of steady motion.

It is thus seen that to change the motion, leaving everything else unchanged, we need careful adjustment of elevator and thrust.

69. **Climbing.**—Suppose then that we make a change in the elevator, the thrust being practically the same as before. In Fig. 36 (*d*) we see that the direction of the propeller axis is changed by the angle wrw', whilst the direction of the pressure is changed by the angle wtw'. In actual machines the latter is very small compared to the former, since the thrust is only a small fraction of the weight, so that wr or $w'r$ is small compared to rt. A small change of the direction of the air pressure in the body means a small change of the relative direction of the velocity. Hence we get a considerable change in the direction of flight. Thus if the elevator is turned down, the direction of steady flight is raised; if the elevator is turned up, the direction of steady flight is lowered. It is this fact that justifies the use of the term "elevator" for the movable part of the horizontal tail. In practice the elevator would be used for changing the angle of attack (§ 67), whilst for climbing additional power would be given by the engine; for a descent the power would be diminished.

70. **Angle of Attack; Lift and Drag.**—Let us now return to the velocity. It will be clear that there is no definite arrangement of elevator and thrust for a given velocity in a given direction, so long as the orientation of the machine is not given, or, which is the same thing, if the direction of the velocity relatively to the machine is not given. If the pilot wishes to fly at a given angular elevation with a given speed, he must choose the direction of the velocity relatively to the machine. To express this mathematically, we define the *angle of attack* as the downward angle between the chord of the wing section and the direction of motion, Fig. 37 (*a*). When this is chosen, the thrust and elevator position are fixed, the latter because the direction of the velocity to the machine giving $G_3 = 0$ is determined by the shape, *i.e.* by the position of the elevator; and the former because it has to balance the air pressure and the weight.

Consider first the case of flight in a horizontal direction with propeller axis horizontal, elevator neutral, which we may call **Normal Flight**. The choice of angle of attack is generally determined by considerations of economy in the use of petrol. The smaller the angle of attack, the smaller is the R_1 component of the air pressure, or the **drag**, as it is called in practical aeronautics. But it is not enough to make R_1 small. We must have the air pressure sufficient to "lift" the machine, *i.e.* to keep it up in the air against the force of gravity. Hence the angle of attack must be arranged

so that for a given vertical component of the air pressure, called in practice the **lift**, the drag shall be as small as possible. The important considerations in practice—for horizontal steady flight—are to have the maximum lift and the maximum ratio lift/drag. Using the symbols L and D (equivalent, in this case, to mR_1 and mR_2), we must have

$$L \quad \text{as large as possible,}$$
$$L/D \text{ as large as possible}$$

Here, then, is a fundamental problem for aerodynamical research, analytical

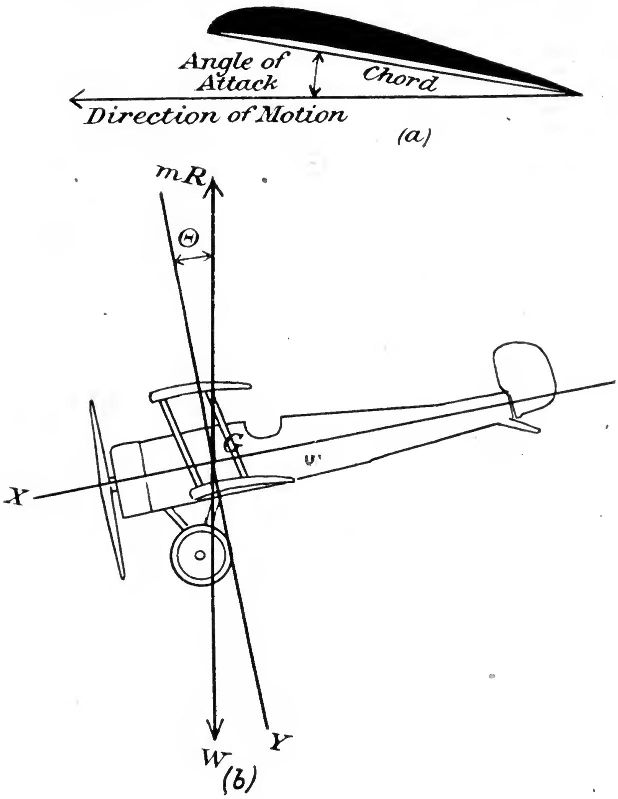

Fig. 37.—(*a*) Chord and Angle of Attack ; (*b*) Glide with no Thrust, Neutral Elevator.

or experimental: to find the form of wing (and body) so that the lift shall be as much as possible when the ratio of lift/drag is as large as possible.

Since a machine is made primarily for the steady flight just considered. this aerodynamical problem is indicative of practical aeroplane research, When once the machine has been made to satisfy these conditions, it is clearly advisable to take advantage of them in any form of steady flight. The most useful angle of attack is thus, more or less, fixed in any given machine, with the elevator in its neutral position. By changing the angle of attack, we make the direction of motion different relatively to the body. This is brought about by manipulation of the elevator and the engine throttle.

There is thus a definite range of possible velocities in horizontal flight, the direction of the propeller axis being different for each velocity. For one particular velocity this axis is horizontal; with smaller velocity, and, therefore, larger angle of attack, the axis points a little upwards; with larger velocity, and, consequently, smaller angle of attack, it points a little downwards. For the maximum angle of attack there is minimum velocity. Since, as shown by experiment, aerofoils have the property that for a certain angle of attack the L/D ratio begins to sink rapidly, the danger of flying too slowly, or of trying to climb too steeply, is obvious. This is called "stalling" the machine.

71. **Glide.**—The amount of propeller thrust required to maintain steady motion depends on the velocity, its direction in space, and its direction relatively to the body. We now take the case in which there there is no propeller thrust at all (see Ch. VII., § 216). The equations for steady motion become

$$R_1 = g \sin \Theta, \quad R_2 = g \cos \Theta, \quad G_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

$G_3 = 0$ defines a certain direction of the velocity relatively to the machine, the elevator being in some given position. The first two equations thus give a definite positive angle Θ (Fig. 37 (*b*)), and the velocity, since R_1 and R_2 depend on the velocity. Thus for a given position of the elevator there is a definite arrangement of steady flight, *i.e.* a definite velocity in a certain direction, the machine having a definite orientation in space. Such a flight is called a *glide*, and for the particular position of the elevator the angle that the direction of motion makes with the horizontal is called the *angle of glide*, or *gliding angle*, for the given elevator position. When the elevator is in its neutral position, we get the angle of glide, Θ, without any control by the pilot. The importance of this case of steady motion is thus made evident. Under the influence of gravity alone the machine will glide down at this *gliding angle* with a certain velocity.

If for any given position of the elevator the pressure makes an angle θ with the Y axis, then $R_1 = R \sin \theta$, $R_2 = R \cos \theta$, and $R = g$. This is, of course, evident *a priori*, since in a glide the air pressure and the gravity are the only two forces acting on the machine, and they must, therefore, balance. The air pressure must be vertical and the gliding angle is $\theta +$ the angle that the velocity makes with the propeller axis. Further, the condition $R = g$ determines the velocity, and the rate of vertical fall is the vertical component of this velocity, *i.e.* the velocity multiplied by the sine of the angle of glide. For safety, this should be as small as possible.

It is thus of great importance to have the angle of glide as small as possible for any given position of the elevator—in particular, for the case when the elevator is in its neutral position. This desideratum is not only useful to keep the rate of fall small, but also advantageous because the pilot then has a greater range in the choice of a landing-place. In a good machine the gliding angle for zero elevator turn should not be more than a few degrees. In practice this means that the ratio lift/drag should be large, another reason for this important condition.

72. **General Problem.**—In all that has gone before, we have assumed what are virtually statical conditions, *i.e.* steady motion in the form of constant velocity in constant direction with constant orientation of the machine. We have considered the changed steady motions produced by changed conditions of thrust and elevator position. But we have not

discussed how the machine goes from one steady motion to another when any change of conditions is introduced. Thus, suppose the machine is flying normally, *i.e.* with horizontal velocity and propeller axis horizontal. We have seen that by using the elevator and the engine throttle we can produce conditions in which the velocity is, say, increased. But when the changes have been made, the inertia of the machine prevents any sudden change from the old velocity of steady motion to the new one The machine is thus in a condition where its appropriate steady motion is different from the motion it actually has just after the introduction of the change. In the same way, more generally, if the machine has a certain type of steady motion, and changes are introduced, the new type of steady motion is different from the actual motion just after the change. Will the machine necessarily tend to assume the new steady motion; and if it does, how does it perform this evolution?

We are here confronted with the general problem of aeroplane motion (restricted by the symmetry and the fact that the vertical plane of motion is the plane of symmetry of the machine). The equations of motion referred to horizontal and vertical axes have already been given, § 65 (77). R_1, R_2, G_3, at any instant, depend on all the circumstances of the motion at the instant. Since the motion of the machine is not steady, the motion produced in the air is not steady, *i.e.* if we get a sort of mental photograph of the positions and motions of all the air particles at one instant, this is not identical with the photograph for the next instant. Thus R_1, R_2, and G_3 really depend on the time, as well as on the velocities of translation and rotation of the machine and on the density of the air. Further, the thrust T is not constant, since it is dependent on the relative motion of the propeller in the air and on the state of motion of the air itself. Expressed analytically, we must say that R_1, R_2, G_3, T are functions of ρ, dx/dt, dy/dt, $d\Theta/dt$, and t. It is quite obvious that a direct attack of the equations of motion is, at least with our present mathematical equipment, quite out of the question.

73. **Simplification of the Problem.**—We at once introduce two simplifications. As in former problems, we ignore any changes in R_1, R_2, G_3, T due to changes in ρ and due to the time. In other words, we take the air to be homogeneous (which is not far wrong if the equations refer to flight at a more or less constant level), and we take R_1, R_2, G_3, and T to have the values they would have if the relative air motion at any instant were a steady motion. The latter assumption is justifiable if the changes in the translational and rotational velocities are not large. We, therefore, make R_1, R_2, G_3, and T depend only on dx/dt, dy/dt, $d\Theta/dt$ (see § 39, Chapter I.).

74. **Moving Axes.**—But the equations of motion (77) are still inconvenient, because of the presence of the factors $\sin\Theta$, $\cos\Theta$ in the first two. To remove them, we must refer the motion to the axes GX, GY, which are fixed in the body. These axes, however, are not fixed in space, and we must first examine carefully what are the accelerations of the centre of gravity G along the directions GX, GY. It is not the same thing as taking the accelerations along the tangent and normal to the path, as was done in § 51 in the case of the phugoids, because the path direction is not along GX.

The theory of moving axes is so fundamental to all aeroplane dynamics that we shall deduce the accelerations from first principles. In

Fig. 38 let GX, GY be the position of the "body axes" at any moment, and $G'X'$, $G'Y'$ at some interval later. Let the velocity components of the body be U_1, U_2 at first, then U_1', U_2' later. Let Θ be the angle that GX makes with the horizontal, Θ' the angle that $G'X'$ makes with the horizontal.

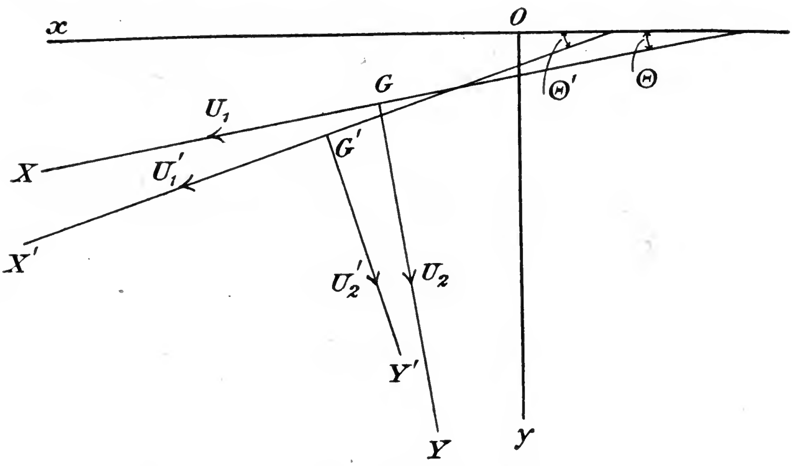

FIG. 38.—Moving Axes in Two-Dimensional Dynamics.

The new velocity components along GX, GY are

$$U_1' \cos(\Theta' - \Theta) - U_2' \sin(\Theta' - \Theta), \quad U_1' \sin(\Theta' - \Theta) + U_2' \cos(\Theta' - \Theta).$$

The changes produced in the velocity components along GX, GY are, therefore,

$$U_1' \cos(\Theta' - \Theta) - U_1 - U_2' \sin(\Theta' - \Theta), \quad U_1' \sin(\Theta' - \Theta) + U_2' \cos(\Theta' - \Theta) - U_2.$$

If the interval is very short so that $\Theta' - \Theta$ is very small, we can, to the first order of small quantities, write the changes in the velocity components as

$$(U_1' - U_1) - U_2(\Theta' - \Theta), \quad U_1(\Theta' - \Theta) + (U_2' - U_2),$$

which means that the components of acceleration are

$$\frac{dU_1}{dt} - U_2 \frac{d\Theta}{dt}, \quad \frac{dU_2}{dt} + U_1 \frac{d\Theta}{dt}. \qquad (80)$$

If U_2 is always zero, so that GX is always tangential to the path, we get

$$\frac{dU_1}{dt}, \quad U_1 \frac{d\Theta}{dt},$$

which are readily seen to agree with the ordinary tangential and normal accelerations.

The equations of motion are now

$$\left.\begin{aligned} \frac{dU_1}{dt} - U_2 \frac{d\Theta}{dt} &= g \sin \Theta + T - R_1, \\ \frac{dU_2}{dt} + U_1 \frac{d\Theta}{dt} &= g \cos \Theta - R_2, \\ \frac{d^2\Theta}{dt^2} &= -G_3. \end{aligned}\right\} \qquad (81)$$

75. **General Problem Insoluble.**—The alternative form of the equations of motion is advantageous not only because R_1, R_2, and T no longer occur with factors $\sin\Theta$, $\cos\Theta$, but also because these quantities are more readily definable in terms of the motion of the machine relatively to itself, rather than relatively to external standards. In the functional forms of the air pressure deduced in Chapter I., §§ 34–36, the quantities in each function are just the velocities and rotation U_1, U_2, $d\Theta/dt$. The case of steady motion is also more easily understandable, and the student should go through the arguments of § 66 with these new equations of motion.

Yet, in spite of all the simplification, we are not, at present, very much nearer solving the motion. R_1, R_2, G_3, and T involve all three velocities, U_1, U_2, $d\Theta/dt$; and even if we knew exactly how these are involved, it would be a difficult (perhaps an impossible) task to unravel them, or, as the mathematicians say, to find an equation for one of them by the elimination of the other two.

The general dynamics of an aeroplane is, therefore, a problem for the future. Perhaps new ideas will occur to mathematicians; or, which is more immediately probable, as in the theory of the Problem of Three Bodies, special solutions and special integrable cases will be found in addition to the steady motion problem.

76. **Small Divergence from Steady Motion; Longitudinal Stability.**—If we refer to § 72, we shall see that for practical purposes it is most important to consider (i) *whether* an aeroplane will tend to assume the steady motion appropriate to the shape and thrust, and (ii), if so, *how* this is done; it is presupposed that the problem arises through a sudden change in the shape and thrust, the air being still "at rest." Now no pilot would make any violent sudden changes; we can then assume that the steady motion for the new arrangement is not very different from the steady motion before the change, *i.e.* the actual initial conditions under the new arrangement.

The practical problem of aeroplane motion is, therefore, as follows: At a given instant an aeroplane's motion is *slightly different* from the steady motion suitable to its configuration and propeller thrust; will the aeroplane tend to assume the steady motion, and how will it do so? When we have solved this problem, we shall then see that more general cases will also admit of partial, if not complete, discussion.

We have, then, to investigate the motion of a slightly disturbed aeroplane in air "at rest." The method to be used must evidently depend upon the principle that if we have a small quantity, then squares and higher powers of this quantity can be neglected in comparison with the quantity itself. Strictly speaking, the investigation will only refer to vanishingly small disturbances, or, rather, divergences from the steady motion. But even for finite, but small, divergences the investigation will give results correct within a certain range of approximation. The problem is that of the *Longitudinal Stability of the Aeroplane.*

The mechanics of the stability of an aeroplane is of a comparatively complicated nature and involves a considerable amount of higher pure mathematics. In order to make the ideas clear and to lead up to the general complication by easy stages, we shall discuss one or two simple problems suggested by our present subject.

77. **Vertical Stability of the Parachute.**—Let us, then, take

first the case of the parachute, which is supposed to be falling vertically with its terminal velocity. A change is made in the parachute, *e.g.* a small body carried by the passenger is dropped. The result is that the parachute is slightly lighter, whereas the air resistance is the same as before. The terminal velocity becomes a little less, and the problem is: A parachute is such that its steady motion is slightly less than its actual motion at a certain instant: will the motion tend to become the steady motion, and how?

The equation of motion of the parachute is (§ 43)

$$\frac{dU}{dt}=\frac{d^2y}{dt^2}=g\left(1-\frac{U^2}{k^2}\right),$$

where U is the velocity at any instant, and k is the new terminal velocity. Initially U is slightly greater than k. Let us then put

$$U=k+u,$$

and let us try whether we are right in assuming that u is small all through the motion, if it is small initially. *We are really carrying out an experiment:* it may be that if we assume u small, we shall arrive at an inconsistent equation. If so, we shall have to relinquish the attempt, and conclude that the assumption is unjustified.

If we put $U=k+u$, the equation of motion becomes

$$\frac{du}{dt}=g\left(1-\frac{k^2+2ku+u^2}{k^2}\right)=-\frac{2g}{k}\left(u+\frac{u^2}{2k}\right).$$

Assuming that u is small during the motion, we neglect $u^2/2k$ compared to u; we get

$$\frac{du}{dt}=-\frac{2g}{k}u,$$

so that

$$u=u_0e^{-\frac{2g}{k}t}, \quad \ldots\ldots\ldots \quad (82)$$

where u_0 is the value of u at $t=0$. Measuring t from the instant when the small weight is dropped, and taking u_0 to be small, we see that u does remain small; in fact, it diminishes and ultimately vanishes—it becomes practically evanescent at quite a short time after the disturbance is introduced. We thus see that the vertical fall of the parachute is stable.

Instead of the weight being dropped, we can suppose some other disturbance which makes the parachute have a velocity slightly different (in excess or in defect) from the steady motion it should have after the disturbance. The above analysis holds whether u_0 is positive or negative. It is also possible to give the parachute such a disturbance that the terminal velocity is unaffected, but the instantaneous velocity is slightly changed. The same result holds: the parachute is stable as regards its vertical fall.

78. **Statical and Dynamical Stability.**—But we must clearly understand what this stability means. In statical problems with bodies at rest, stability means that any disturbance introduced tends to be wiped out, leaving ultimately no trace of its ever having existed. It is not so in the case of the parachute. The disturbance has its effect, which could always be measured or observed, if sufficiently delicate apparatus were used.

Let us find the distance the parachute in § 77 falls in a time T after the disturbance. This distance is

$$\int_0^T U\,dt = \int_0^T (k+u)\,dt = kT + u_0 \int_0^T e^{-\frac{2g}{k}t}\,dt$$
$$= kT + \frac{u_0 k}{2g}\left(1 - e^{-\frac{2g}{k}T}\right).$$

The distance it would have fallen is, of course, kT. Hence the disturbance produces the result that, at any time T after the disturbance, the parachute is a distance $\frac{u_0 k}{2g}\left(1 - e^{-\frac{2g}{k}T}\right)$ lower than it would have been if it had moved with the appropriate steady motion. *The effect of the disturbance thus persists indefinitely:* ultimately it is measured by the distance $u_0 k/2g$.

This is an important difference between statical and dynamical stability. In the former the effect of the disturbance disappears altogether after a sufficient interval. In the latter the effect persists. But we are justified in using the term "stable," in reference to the dynamical problem, because, both theoretically and practically, the important thing about the parachute is its velocity when it reaches the ground, and not whether it reaches the ground a fraction of a second before or after it is expected to do so.

79. **Longitudinal Stability of the Parachute.**—We have examined the stability of the parachute on the assumption that it is always moving in the same vertical straight line. Now, actually the parachute undergoes all kinds of disturbances, and we must consider the more general problem. Let us, then, suppose that a disturbance of the most general kind is introduced, with the restriction that motion is still to be in one vertical plane, or two-dimensional.

To imagine such a case, let the passenger throw away the small weight mentioned in § 77. If he projects it in some direction inclined to the vertical, then the backward impulse will produce a small change in the vertical velocity at the same time as a small horizontal velocity and a small rotation are communicated to the parachute. We have, then, the following problem: A parachute (whose terminal velocity k can be found, or is given) does not fall steadily with this velocity, but has, at some instant, a slightly different vertical velocity, a small horizontal velocity, and a small angular velocity. Will the parachute tend to assume the steady motion, *i.e.* will the vertical velocity tend to become k, and will the horizontal velocity and the rotation tend to disappear? Or will the slight divergence from the terminal motion tend to increase indefinitely, so that, *e.g.*, the parachute will turn over and over in its fall? This problem is of great practical significance, as we shall find that a parachute is only stable if certain conditions are satisfied; otherwise it is an unsafe vehicle of descent, and may lead to disaster.

Using the axes of Fig. 39, let the centre of gravity G be taken as origin of a pair of axes GX, GY fixed in the parachute, and lying in the vertical plane of motion, such that when G is at the origin O, and GX coincides with Ox, then GY coincides with Oy. Let Θ be the angle between GX and Ox. We choose GX so that in steady fall it is

horizontal and GY vertical. If we use GX, GY as moving axes, the acceleration components along GX, GY are

$$\frac{dU_1}{dt} - U_2\frac{d\Theta}{dt}, \quad \frac{dU_2}{dt} + U_1\frac{d\Theta}{dt},$$

where U_1, U_2 are the velocities along GX, GY. Let the air resistance give forces R_1, R_2 per unit mass and moment G_3 per unit moment of inertia from Y to X, as shown in the figure. The weight is mg, m being the mass. Then we have

$$\left.\begin{aligned} \frac{dU_1}{dt} - U_2\frac{d\Theta}{dt} &= g\sin\Theta - R_1, \\ \frac{dU_2}{dt} + U_1\frac{d\Theta}{dt} &= g\cos\Theta - R_2, \\ \frac{d^2\Theta}{dt^2} &= -G_3. \end{aligned}\right\} \quad \ldots \quad (83)$$

As before, R_1, R_2, G_3 depend on U_1, U_2, $d\Theta/dt$.

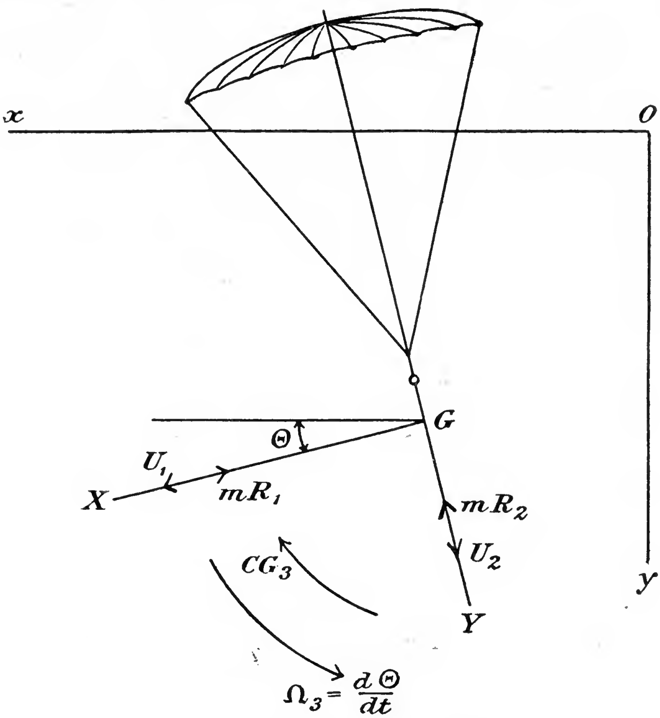

FIG. 39.—Parachute: Longitudinal Stability.

Now in steady fall we have

$$U_1 = \frac{dU_1}{dt} = \frac{dU_2}{dt} = \Theta = \frac{d\Theta}{dt} = \frac{d^2\Theta}{dt^2} = 0.$$

If, then, R_{10}, R_{20}, G_{30} are the values of R_1, R_2, G_3 for steady fall, we clearly have $R_{10} = 0$, $R_{20} = g$, $G_{30} = 0$. In the actual motion we have $U_1 = u_1$, $U_2 = k + u_2$, $\Theta = \theta, d\Theta/dt = \Omega_3 = d\theta/dt = \omega_3$, where u_1, u_2, θ, ω_3

are all small initially. Let us make the experiment of trying whether we get consistent results by assuming that u_1, u_2, θ, ω_3 are small all through the motion. We, therefore, attempt to ignore squares and products of these quantities.

80. **The Derivatives for the Parachute.**—Now R_1, R_2, G_3 are functions of U_1, U_2, $d\Theta/dt$, *i.e.* of u_1, $k+u_2$, ω_3. We thus have by Chapter I., § 37,

$$\begin{aligned} R_1 &= \quad k(a_x u_1 + b_x u_2 + f_x \omega_3), \\ R_2 &= g + k(a_y u_1 + b_y u_2 + f_y \omega_3), \\ G_3 &= \quad k(a_3 u_1 + b_3 u_2 + f_3 \omega_3), \end{aligned}$$

where the derivatives a_x, b_x, etc., are constants which depend on the parachute and its steady motion, and not on the disturbance itself or its effects. Only first powers of u_1, u_2, ω_3 are used, since we are trying whether u_1, u_2, ω_3 do really remain small during the motion.

Before we substitute the values of R_1, R_2, G_3 in the equations of motion, we shall show that certain derivatives are really zero. Suppose u_2 and ω_3 are zero. Then $ku_1 a_x$ is the additional resistance due to a small velocity u_1. Such a force exists *a priori*. But let u_1 and ω_3 be zero; $ku_2 b_x$ is a small horizontal force due to a change in the downward vertical fall of a parachute. But the parachute is symmetrical about the axis of Y. Hence $b_x = 0$, and R_1 involves u_2 only in squares, etc. f_x is not zero. Again, in R_2 let $u_2 = \omega_3 = 0$. Then $ku_1 a_y$ is the additional R_2 resistance due to a small velocity u_1. If a_y exists, then a negative u_1 would give an opposite effect to a positive u_1, and this is impossible by the symmetry. Hence $a_y = 0$, and R_2 involves u_1 only in squares, etc. Similarly, $f_y = 0$; but b_y is not zero. Finally, b_3 must be zero by symmetry. We thus get

$$R_1 = k(a_x u_1 + f_x \omega_3), \quad R_2 = g + kb_y u_2, \quad G_3 = k(a_3 u_1 + f_3 \omega_3).$$

81. **Solution of the Equations of Motion for Assumed Small Divergence from Steady Motion.**—To the first order of small quantities the equations of motion are now

$$\left.\begin{aligned} \frac{du_1}{dt} - k\frac{d\theta}{dt} &= g\theta - k\left(a_x u_1 + f_x \frac{d\theta}{dt}\right), \\ \frac{du_2}{dt} &= -kb_y u_2, \\ \frac{d^2\theta}{dt^2} &= -k\left(a_3 u_1 + f_3 \frac{d\theta}{dt}\right). \end{aligned}\right\} \quad \cdots\cdots \quad (84)$$

Introduce the symbol $D \equiv d/dt$. We get for the first and third equations

$$\left.\begin{aligned} (D + ka_x)u_1 + (\overline{f_x - 1} \,.\, kD - g)\theta = 0, \\ (\quad ka_3)u_1 + (D^2 + kf_3 D)\theta = 0. \end{aligned}\right\} \quad \cdots\cdots \quad (85)$$

The second equation (84) gives at once

$$u_2 = u_{20} e^{-kb_y t},$$

where u_{20} is the value of u_2 at $t = 0$. Now b_y is positive *a priori*, without any aerodynamics, since an increased downward fall must give increased upward pressure, and *vice versa*. There is, thus, no inconsistency so far as the disturbance in the vertical velocity is concerned. If u_{20} is small initially, u_2 remains small always. This agrees with the result of § 77.

We now have to consider the first and third equations. We can easily eliminate u_1 from the two equations (85). We get for θ the equation

$$\{(D+ka_x)(D^2+kf_3D)-ka_3(\overline{f_x-1}\,.\,kD-g)\}\,\theta=0,$$

a linear differential equation of the third order with constant coefficients: it is

$$\{D^3+k(a_x+f_3)D^2+k^2(a_xf_3-a_3f_x+a_3)D+kga_3\}\,\theta=0.$$

To solve, let us suppose that

$$\theta=C_1e^{k\lambda_1t}+C_2e^{k\lambda_2t}+\ldots$$

Substitute: we get

$$C_1\{\lambda_1^3+(a_x+f_3)\lambda_1^2+(a_xf_3-a_3f_x+a_3)\lambda_1+\frac{g}{k^2}a_3\}\,e^{k\lambda_1t}+\ldots=0.$$

This is true for all values of t. Hence λ_1 satisfies the equation

$$\lambda_1^3+(a_x+f_3)\lambda_1^2+(a_xf_3-a_3f_x+a_3)\lambda_1+\frac{g}{k^2}a_3=0.$$

Similarly, λ_2 satisfies this equation with λ_2 instead of λ_1; and so on. Hence there are *three* values of λ, and these are given by the algebraic equation

$$\lambda^3+(a_x+f_3)\lambda^2+(a_xf_3-a_3f_x+a_3)\lambda+\frac{g}{k^2}a_3=0 \quad \ldots \quad (86)$$

If λ_1, λ_2, λ_3 are the solutions, we have

$$\theta=C_1e^{k\lambda_1t}+C_2e^{k\lambda_2t}+C_3e^{k\lambda_3t};$$

then by the second equation (85) we get

$$-a_3\,.\,u_1=kC_1(\lambda_1^2+f_3\lambda_1)\,e^{k\lambda_1t}+kC_2(\lambda_2^2+f_3\lambda_2)\,e^{k\lambda_2t}+kC_3(\lambda_3^2+f_3\lambda_3)\,e^{k\lambda_3t}.$$

The three quantities C_1, C_2, C_3 are, algebraically speaking, arbitrary. In the definite problem in hand they are determined by the initial conditions, which are (i) the value of u_1, (ii) the value of θ, (iii) the value of ω_3, all at $t=0$. Call these u_{10}, θ_0, ω_{30}. Then we have, with a little reduction,

$$\left.\begin{aligned}C_1+C_2+C_3=\theta_0,\quad \lambda_1C_1+\lambda_2C_2+\lambda_3C_3&=\frac{\omega_{30}}{k},\\ \lambda_1^2C_1+\lambda_2^2C_2+\lambda_3^2C_3&=\frac{-a_3u_{10}-f_3\omega_{30}}{k}.\end{aligned}\right\} \quad \ldots \quad (87)$$

The motion is thus solved, *temporarily*.

82. **Stability Conditions; Theory of Equations.**—We say temporarily, because it is all based on the assumption that u_1, θ, ω_3 are small throughout the motion, if u_{10}, θ_0, ω_{30} are small. But if u_{10}, θ_0, ω_{30} are small, then C_1, C_2, C_3 must be small. Everything then depends on the exponentials $e^{k\lambda_1t}$, $e^{k\lambda_2t}$, $e^{k\lambda_3t}$; these must be such that they do not increase indefinitely, so that if λ_1, λ_2, λ_3 are real, they must be negative (at least, not positive), and if (two of them are) complex, the real parts must be negative (or not positive).

The stability of the parachute thus depends on whether the solutions of the algebraic equation (86) have their real parts *all* negative (including the possibility that the roots are all real). The problem, then, reduces to a discussion of the signs of the real parts of the roots of a cubic equation.

The general method for such a discussion is somewhat difficult, and will be found in Routh's *Rigid Dynamics*, Part II., p. 238. We shall here adopt a simplified *ad hoc* process which will suffice for our purposes.

Consider the *quadratic* equation

$$\lambda^2 + a\lambda + b = 0 \quad \dots\dots\dots\dots \quad (88)$$

If this is to have the roots negative if real, and real parts negative if complex, then a, b must be both positive; these are necessary and sufficient conditions. For we have

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If the roots are real, *i.e.* $a^2 > 4b$, they will be both negative if a is positive and $\sqrt{a^2 - 4b} < a$, which means b positive. If the roots are complex, so that $a^2 < 4b$, then a must be positive to make the real parts of the roots negative, and, of course, b is positive.

Now take the *cubic* equation

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0. \quad \dots\dots\dots \quad (89)$$

One root must be real, and it is negative if C is positive; let it be $-c$. Hence the equation can be written

$$\lambda^3 + A\lambda^2 + B\lambda + C \equiv (\lambda^2 + a\lambda + b)(\lambda + c) = 0. \quad \dots\dots \quad (90)$$

To make all the real parts negative, we must have a, b positive, and c is positive. Thus

$$A = a + c, \quad B = b + ac, \quad C = bc, \quad \dots\dots\dots \quad (91)$$

where a, b, c are all positive. Hence it is *necessary* for A, B, C to be all positive. But are these conditions *sufficient*? We can soon see that this is not so. For, suppose that the equation has two complex roots, and let the real parts of these complex roots vanish, so that we have two roots $\pm\iota\mu$. We get for μ the equation

$$-\iota(\mu^3 - B\mu) - (A\mu^2 - C) = 0,$$

which means $\mu^3 - B\mu = 0$, $A\mu^2 - C = 0$. We cannot have $\mu = 0$, since C is not zero. Hence we get $\mu^2 = B = C/A$, so that $AB - C = 0$. Thus, if A, B, C are all positive, but it happens that $AB = C$, we get the real parts vanishing. It is, therefore, clear that some further condition or conditions must be found in order to get the *sufficient* conditions.

Consider the quantity $AB - C$. In terms of a, b, c, it is by (91)

$$a(c^2 + ac + b).$$

Since a, b, c must be positive, it follows that $AB - C$ must be positive. We shall now see that if A, B, C are positive, and $AB - C$ is positive, then a, b, c are positive.

We have c positive, so that as $bc = C$ and C is positive, b is positive; also, as $c^2 + ac = c(a + c) = cA$ and A is positive, $c^2 + ac$ is positive. It follows that $c^2 + ac + b$ is positive, and, therefore, since $a(c^2 + ac + b)$ is positive, a is positive.

We, therefore, conclude that the equation $\lambda^3 + A\lambda^2 + B\lambda + C = 0$ has the real parts of its roots all negative (including the case where all the roots are real) if, and only if,

$$A, \ B, \ C, \ AB - C. \quad \dots\dots\dots\dots \quad (92)$$

are all positive. It is unnecessary to specify B positive, since it is implied in the last condition. The necessary and sufficient conditions are A, $AB - C$, C all positive.

83. **Conditions of Stability of the Parachute.**—In our parachute problem we have

$$A = a_x + f_3,\ B = a_x f_3 - a_3 f_x + a_3,\ C = \frac{g}{k^2} a_3.$$

Hence the assumption of stability is justified if we have

$$\left.\begin{aligned} &a_x + f_3 \text{ positive, } a_3 \text{ positive, and} \\ &(a_x + f_3)(a_x f_3 - a_3 f_x + a_3) - \frac{g}{k^2} a_3 \text{ positive.} \end{aligned}\right\} \quad \ldots \quad (93)$$

84. We have thus found the conditions of stability for a parachute, assuming that the motion is in a vertical plane, and that it is symmetrical about an axis in this plane. The quantities in (93) must be found by aerodynamics; we shall return to this later, Chapter VII. Meanwhile, we note that if these conditions are satisfied, u_1, u_2, θ, ω_3 remain small all through the motion, and we have an approximate solution. If, however, the conditions are not satisfied, u_1, u_2, θ, ω_3 will not remain small, and we have no solution at all—in fact, all the work is wasted, except for the information it gives us that the assumption u_1, u_2, θ, ω_3 small, is unjustified.

It is instructive to note that it is sufficient for one of the λ's to have its real part positive in order to upset the stability. Stability requires a number of conditions, which must all be satisfied simultaneously. We have seen that on the assumption u_1, u_2, θ, ω_3 small, we have no inconsistency for u_2. This would indicate that the instability, if the parachute is unstable, arises from a tendency to overturn rather than from a tendency to come down with a rush. Such interpretations are, however, very risky, since the result $u_2 = u_{20} e^{-kb_y t}$ itself depends on the assumption u_1, u_2, θ, ω_3 small, which, in the case of instability, is not true.

It is now a matter of algebra to write down the actual motion performed by the parachute in its tendency to assume the appropriate steady motion, given the values of the quantities u_1, u_2, θ, ω_3 at the time $t = 0$. We solve the equation (86), substitute in equations (87), and solve for C_1, C_2, C_3. The process is very laborious, and in the case of the parachute the result is of no particular value.

85. **Longitudinal Stability of the Aeroplane in Gliding Flight.**—We shall now apply the method of §§ 77, etc., to the case of the aeroplane. To envisage a practical problem, let us consider what happens if the machine is flying steadily and horizontally, elevator in some given position, and the engine is so arranged that there is no thrust. The machine must fall, but the question is: How will it fall? We have seen that with no thrust there is a certain type of steady motion in the form of a glide. The actual motion is different. Will the machine tend to settle down to the glide, or will it move in some entirely different way?

In practical flying, the gliding angle is so small that we may consider the initial conditions, *i.e.* the motion at the time $t = 0$ measured from the instant when the engine is altered, to be only slightly different from the gliding motion. Let us now make the assumption that this divergence remains small; if we get equations consistent with this assumption, then the machine will tend to settle down into the gliding motion, and these equations will actually represent the motion. But if, on this assumption,

we get equations inconsistent with it, it will mean that the machine does not tend to assume the gliding motion, and that the equations are of no value in indicating the motion the machine actually does execute (except for only a very short interval of time).

In the gliding flight let the velocities be U_{10}, U_{20}, and let Θ_0 be the angle that the propeller axis makes with the horizontal during the glide. In the actual motion let

$$U_1 = U_{10} + u_1,\ \ U_2 = U_{20} + u_2,\ \ \Theta = \Theta_0 + \theta,\ \ d\Theta/dt = \omega_3,$$

where u_1, u_2, θ, ω_3 are small. In the equation of motion (81) we have $T=0$, and, putting $U = (U_{10}^2 + U_{20}^2)^{\frac{1}{2}}$,

$$\begin{aligned} R_1 &= R_{10} + U(a_x u_1 + b_x u_2 + f_x \omega_3), \\ R_2 &= R_{20} + U(a_y u_1 + b_y u_2 + f_y \omega_3), \\ G_3 &= G_{30} + U(a_3 u_1 + b_3 u_2 + f_3 \omega_3). \end{aligned}$$

In the glide we have

$$g \sin \Theta_0 = R_{10},\ \ g \cos \Theta_0 = R_{20},\ \ G_{30} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (94)$$

To the first order, the equations (81) become

$$\left.\begin{aligned} \frac{du_1}{dt} - U_{20}\frac{d\theta}{dt} &= g\theta\cos\Theta_0 - U(a_x u_1 + b_x u_2 + f_x\omega_3), \\ \frac{du_2}{dt} + U_{10}\frac{d\theta}{dt} &= -g\theta\sin\Theta_0 - U(a_y u_1 + b_y u_2 + f_y\omega_3), \\ \frac{d^2\theta}{dt^2} &= - U(a_3 u_1 + b_3 u_2 + f_3\omega_3), \end{aligned}\right\} \quad . \quad . \quad . \qquad (95)$$

where a_x, b_x, etc., are constants which depend on the machine (including the fact that the elevator is in a certain position), and not on the actual motion.

There is now no reason for supposing that any of these are zero. Remembering that $\omega_3 = d\theta/dt$, we have

$$\left.\begin{aligned} &(D + Ua_x)u_1 + (Ub_x)u_2 + (\overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0)\theta = 0, \\ &(Ua_y)u_1 + (D + Ub_y)u_2 + (\overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0)\theta = 0, \\ &(Ua_3)u_1 + (Ub_3)u_2 + (D^2 + Uf_3\,.\,D)\theta = 0, \end{aligned}\right\}. \qquad (96)$$

where D represents the operation d/dt. We must obtain an equation for each one of the quantities u_1, u_2, θ, *i.e.* we must eliminate two of them, say u_2, θ, so as to get an equation for the remaining one, u_1.

The elimination is now more difficult than in § 81. It can be performed, but it is simpler to present the argument as follows. Let us assume

$$\begin{aligned} u_1 &= A_1 e^{U\lambda_1 t} + A_2 e^{U\lambda_2 t} + \ldots = \Sigma A_1 e^{U\lambda_1 t}, \\ u_2 &= B_1 e^{U\lambda_1 t} + B_2 e^{U\lambda_2 t} + \ldots = \Sigma B_1 e^{U\lambda_1 t}, \\ \theta &= C_1 e^{U\lambda_1 t} + C_2 e^{U\lambda_2 t} + \ldots = \Sigma C_1 e^{U\lambda_1 t}, \end{aligned}$$

where λ_1, λ_2, . . . are numbers yet to be found, and A_1, A_2, . . . are constants which will depend on the initial conditions. Substitute in equations (96): we get

$$\begin{aligned} &\Sigma\{(U\lambda_1 + Ua_x)A_1 + (Ub_x)B_1 + (\overline{U^2 f_x - UU_{20}}\,.\,\lambda_1 - g\cos\Theta_0)C_1\}e^{U\lambda_1 t} = 0, \\ &\Sigma\{(Ua_y)A_1 + (U\lambda_1 + Ub_y)B_1 + (\overline{U^2 f_y + UU_{10}}\,.\,\lambda_1 + g\sin\Theta_0)C_1\}e^{U\lambda_1 t} = 0, \\ &\Sigma\{(Ua_3)A_1 + (Ub_3)B_1 + (U^2\lambda_1^2 + U^2 f_3\,.\,\lambda_1)C_1\}e^{U\lambda_1 t} = 0, \end{aligned}$$

for all values of t. Hence λ_1 must satisfy the equations

$$(\lambda_1 + a_x)A_1 + (\quad b_x)B_1 + \left(f_x - \frac{U_{20}}{U}\cdot\lambda_1 - \frac{g}{U^2}\cos\Theta_0\right)UC_1 = 0,$$

$$(\quad a_y)A_1 + (\lambda_1 + b_y)B_1 + \left(f_y + \frac{U_{10}}{U}\cdot\lambda_1 + \frac{g}{U^2}\sin\Theta_0\right)UC_1 = 0,$$

$$(\quad a_3)A_1 + (\quad b_3)B_1 + (\quad \lambda_1^2 + f_3\lambda_1 \quad)UC_1 = 0,$$

simultaneously. Similarly for λ_2, etc. There are, therefore, *four* values of λ, given by the determinantal equation

$$\begin{vmatrix} \lambda + a_x, & b_x, & \left(f_x - \frac{U_{20}}{U}\right)\lambda - \frac{g}{U^2}\cos\Theta_0 \\ a_y, & \lambda + b_y, & \left(f_y + \frac{U_{10}}{U}\right)\lambda + \frac{g}{U^2}\sin\Theta_0 \\ a_3, & b_3, & \lambda^2 + f_3\lambda \end{vmatrix} = 0, \quad . \quad . \quad . \quad (97)$$

and we have

$$\left.\begin{aligned} u_1 &= A_1e^{U\lambda_1 t} + A_2e^{U\lambda_2 t} + A_3e^{U\lambda_3 t} + A_4e^{U\lambda_4 t}, \\ u_2 &= B_1e^{U\lambda_1 t} + B_2e^{U\lambda_2 t} + B_3e^{U\lambda_3 t} + B_4e^{U\lambda_4 t}, \\ \theta &= C_1e^{U\lambda_1 t} + C_2e^{U\lambda_2 t} + C_3e^{U\lambda_3 t} + C_4e^{U\lambda_4 t}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

where λ_1, λ_2, λ_3, λ_4 are the solutions of the algebraic equation (97), and A_1, A_2, . . . are constants which depend on the initial conditions. There are *four* initial conditions, namely, the values at $t = 0$ of u_1, u_2, θ, $w_3\,(= d\theta/dt)$. But equations (96) show that A_1, B_1, C_1 are not independent, that, in fact, if A_1 is given, B_1, C_1 are found from any two of (96). Similar considerations hold for A_2, B_2, C_2, etc. Hence, with given initial conditions, all the twelve constants are known, and we have the complete solution of the problem. We have ignored derivatives due to the propeller.

86. **Conditions of Stability.**—The solution depends on the assumption that u_1, u_2, θ, $d\theta/dt$ remain small all through the motion. Hence our results are only true if it can be shown that this is the case, *i.e.* that the exponential functions $e^{U\lambda_1 t}$, $e^{U\lambda_2 t}$, $e^{U\lambda_3 t}$, $e^{U\lambda_4 t}$ do not increase indefinitely. We must then have the real parts of the roots of the equation (97) all negative (at least, not positive). If any one of the real parts is positive, the whole analysis breaks down, and we have the result that the machine does *not* tend to assume the gliding motion; we know nothing more than this (except for a very short time). We have now to consider the algebraic equation (97), which we write in the form

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

The student will have no difficulty in finding the values of A_1, B_1, C_1, D_1, E_1 (we have, in fact, $A_1 = 1$, but B_1, C_1, D_1, E_1 are comparatively complicated expressions).

Using the fact that $A_1 = 1$ always, let us consider the equation

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

If the roots are all real, we can obviously write

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 \equiv (\lambda^2 + a_1\lambda + b_1)(\lambda^2 + a_2\lambda + b_2),$$

where a_1, b_1, a_2, b_2 are real quantities. If the four roots, or two of them, are complex, we can take each pair of conjugate complex roots together,

and we again get two quadratic factors with a_1, b_1, a_2, b_2 real. Hence the equation can always be written

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 \equiv (\lambda^2 + a_1\lambda + b_1)(\lambda^2 + a_2\lambda + b_2) = 0, \quad . \quad . \quad (101)$$

where a_1, b_1, a_2, b_2 are all real quantities.

If the roots of the biquadratic equation (100) are to have all their real parts negative, the same must be true of the two roots given by each quadratic factor in (101). Hence, as shown above, § 82, it is necessary and sufficient that a_1, b_1, a_2, b_2 shall all be positive. We get at once that B_1, C_1, D_1, E_1 must all be positive.

The conditions are *necessary*. We can show that they are not sufficient by considering when it is possible for the real parts of a pair of conjugate complex roots to be zero. This happens if we have two roots $\pm \iota\mu$, where μ is given by the equation

$$\mu^4 - \iota B_1\mu^3 - C_1\mu^2 + \iota D_1\mu + E_1 = 0,$$

i.e. by the equations $\mu^4 - C_1\mu^2 + E_1 = 0$, and $\mu(B_1\mu^2 - D_1) = 0$. We can discard the case $\mu = 0$, since E_1 does not vanish. Hence we get two roots $\pm \iota\mu$ if we get the condition given by substituting $\mu^2 = D_1/B_1$ in $\mu^4 - C_1\mu^2 + E_1 = 0$. This condition is at once found to be

$$B_1C_1D_1 - D_1^2 - E_1B_1^2 = 0.$$

Thus, even if B_1, C_1, D_1, E_1 are all positive, it is possible to get the real parts of two roots zero, if it happens that

$$B_1C_1D_1 - D_1^2 - E_1B_1^2 = 0.$$

It is clear that we must have more conditions in general, and that these are connected with the expression

$$B_1C_1D_1 - D_1^2 - E_1B_1^2 \equiv H_1.$$

We shall now show that B_1, C_1, D_1, E_1, all positive, and in addition $H_1 \equiv B_1C_1D_1 - D_1^2 - E_1B_1^2$, positive, are the necessary and sufficient conditions that the real parts of the roots of (100) shall be negative, *i.e.* that a_1, b_1, a_2, b_2 in (101) shall be positive. For, consider the expression H_1 in terms of a_1, b_1, a_2, b_2. We have

$$B_1 = a_1 + a_2, \quad C_1 = a_1a_2 + b_1 + b_2, \quad D_1 = a_1b_2 + a_2b_1, \quad E_1 = b_1b_2.$$

Hence

$$H_1 \equiv B_1C_1D_1 - D_1^2 - E_1B_1^2 = a_1a_2[(a_1 + a_2)(a_1b_2 + a_2b_1) + (b_1 - b_2)^2],$$

so that H_1 positive is a *necessary* condition. Conversely, if H_1 is positive, we have a_1a_2 positive, since the factor in square brackets is equal to $B_1C_1 + (b_1 - b_2)^2$, which is essentially positive, as B_1, C_1 are already given positive. We have then a_1a_2 positive, and $a_1 + a_2 (\equiv B_1)$ positive; hence a_1, a_2 are each positive. But we have $D_1 \equiv a_1b_2 + a_2b_1$ positive, and $E_1 \equiv b_1b_2$ positive, where a_1, a_2 are positive; it follows that b_1, b_2 are both positive.

We have, therefore, proved that the necessary and sufficient conditions that the algebraic equation

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0$$

shall have the real parts of its roots all negative is that

$$B_1, \; C_1, \; D_1, \; E_1, \; \text{and} \; H_1 \equiv B_1C_1D_1 - D_1^2 - E_1B_1^2 \quad . \quad . \quad . \quad (102)$$

must all be positive. If the equation is

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

where A_1 is not necessarily unity, the necessary and sufficient conditions (as the student can at once verify) are that

$$A_1,\ B_1,\ C_1,\ D_1,\ E_1, \text{ and } H_1 \equiv B_1C_1D_1 - A_1D_1^2 - E_1B_1^2 \quad . \quad . \quad (103)$$

shall all have the same sign. The quantity H_1 is called Routh's Discriminant.

If, now, these conditions are satisfied, the machine will tend to assume the steady motion appropriate to it, namely, the glide. The way in which this is done is defined by the initial conditions. In the case now considered these are obtained by comparing the steady motion and orientation before the engine is altered with the steady motion and orientation after it is altered. It is clear that to work out the motion explicitly, even on the assumption on which the investigation is based, must be exceedingly laborious, and, in any case, we must first know the values of the derivatives, etc. But whether we work out the motion or not, we are sure that the machine must sooner or later move with practically its gliding motion, and for ordinary purposes this is sufficient.

Instead of the initial conditions suggested here, we can have any other initial conditions, so long as they are not too divergent from the gliding motion.

87. **Longitudinal Stability of the Aeroplane in Steady Motion (General).**—Proceeding now to the general problem, suppose that we know the steady motion appropriate to a machine with given position of elevator and given condition of the engine. If the motion is not actually this steady motion, we must investigate whether the machine will or will not tend to assume the steady motion. *E.g.* we can suppose that the machine has been moving steadily in some way, and then changes are made in the elevator and engine; or we can suppose that in some other way the actual motion at any instant is not quite what it should be for steady motion. The general method is as follows:

In the equations of motion (81) let U_{10}, U_{20}, Θ_0, T_0, R_{10}, R_{20}, G_{30} refer to the steady motion which we want the machine to assume. In the actual motion at any time let the velocity components U_1, U_2 and angle of inclination Θ be

$$U_{10} + u_1,\quad U_{20} + u_2,\quad \Theta_0 + \theta,$$

and let the changes in the air forces and couple (*including the effect on the propeller*) be

$$\begin{gathered} U(a_x u_1 + b_x u_2 + f_x \omega_3), \\ U(a_y u_1 + b_y u_2 + f_y \omega_3), \\ U(a_3 u_1 + b_3 u_2 + f_3 \omega_3), \end{gathered}$$

where $U = (U_{10}{}^2 + U_{20}{}^2)^{\frac{1}{2}}$, and u_1, u_2, θ (and, therefore, also $\omega_3 = d\theta/dt$) are small. The equations of motion, to the first order, are

$$\left.\begin{aligned} \frac{du_1}{dt} - U_{20}\frac{d\theta}{dt} &= g\theta\cos\Theta_0 - U(a_x u_1 + b_x u_2 + f_x\omega_3), \\ \frac{du_2}{dt} + U_{10}\frac{d\theta}{dt} &= -g\theta\sin\Theta_0 - U(a_y u_1 + b_y u_2 + f_y\omega_3), \\ \frac{d^2\theta}{dt^2} &= \qquad\qquad - U(a_3 u_1 + b_3 u_2 + f_3\omega_3), \end{aligned}\right\} \quad . \quad . \quad . \quad (104)$$

since in the steady motion

$$g \sin \Theta_0 = R_{10} - T_0, \quad g \cos \Theta_0 = R_{20}, \quad G_{30} = 0. \quad . \quad . \quad . \quad . \quad (105)$$

The quantities a_x, b_x, etc., are constants depending on the machine, including the position of the elevator, the condition of the engine, etc. But during the motion we are now considering, all these derivatives can be looked upon as given. We see that the equations now obtained are of exactly the same form as in the case of the glide. The stability depends on the roots of the algebraic equation

$$\begin{vmatrix} \lambda + a_x, & b_x, & \left(f_x - \frac{U_{20}}{U}\right)\lambda - \frac{g}{U^2}\cos\Theta_0 \\ a_y, & \lambda + b_y, & \left(f_y + \frac{U_{10}}{U}\right)\lambda + \frac{g}{U^2}\sin\Theta_0 \\ a_3, & b_3, & \lambda^2 + f_3\lambda \end{vmatrix} = 0. \quad . \quad . \quad (106)$$

The real parts of the four solutions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ must be negative (at least, not positive). The conditions for stability are, therefore, as before, that

$$A_1, \ B_1, \ C_1, \ D_1, \ E_1, \ H_1 \equiv B_1C_1D_1 - A_1D_1^2 - E_1B_1^2 \quad . \quad . \quad . \quad (107)$$

all have the same sign, where the above equation is written in the form

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

A_1 is again unity; the other coefficients can be easily calculated.

We have thus found the conditions for the stability of any form of steady flight in longitudinal motion, *i.e.* whether the machine will tend to assume this steady flight if it is at any moment moving in a slightly different manner. In any given case we have but to find the quantities

$$a_x, \ b_x, \ f_x, \ a_y, \ b_y, \ f_y, \ a_3, \ b_3, \ f_3,$$

and to substitute in the conditions.

88. **Stability Depends on the Steady Motion.**—It should be remembered that *the derivatives* are all taken *to refer to the particular steady motion dealt with.* Thus a_x is not the same for all glides or for all horizontal flights. This means that the stability of a machine is not a constant property, but one that depends on the kind of steady flight appropriate to it in the conditions of its elevator and engine. It does not follow that an aeroplane is stable for all steady motions because it happens to be stable for a particular steady motion.

In practice the most important steady motion is the horizontal one with neutral position of elevator. We put $\Theta_0 = 0$ in (106). Machines are designed for stability in this case. Since the stability conditions are not equations, but rather inequalities, *i.e.*

$$B_1 > 0, \ C_1 > 0, \ D_1 > 0, \ E_1 > 0, \ B_1C_1D_1 - D_1^2 - E_1B_1^2 > 0,$$

we have a *range of stability*, *i.e.* a machine can be made stable for a range of steady motions, as, *e.g.*, for all steady flights in a horizontal direction (the propeller axis being horizontal or not) with the speed lying between certain limits, and for steady flights not far removed from the horizontal direction, including gliding.

89. **More General Problem; Vertical Dive.**—It has been said that the general motion of a given aeroplane cannot at present be discussed. We have, so far, dealt with steady motions and their stability, *i.e.* oscillations about steady motions. One can readily imagine cases of aeroplane motion that should lead to interesting problems. One very important case is that of the *vertical dive.*

So long as a machine is to move in a straight line inclined to the vertical, say in a horizontal direction, it is necessary to restrict the motion to that of steady motion so as to have a statical problem. To prove this, let us see what is required to produce a rectilinear acceleration. Let the figure represent the forces acting on the aeroplane, U being the velocity at any moment, the path being assumed straight, so that we have rectilinear acceleration. The resultant of mT, W, mR must then be in the direction of the path. Now W is absolutely constant, whilst mT and mR are fixed in direction, but vary in amount as the velocity varies. Draw rt parallel

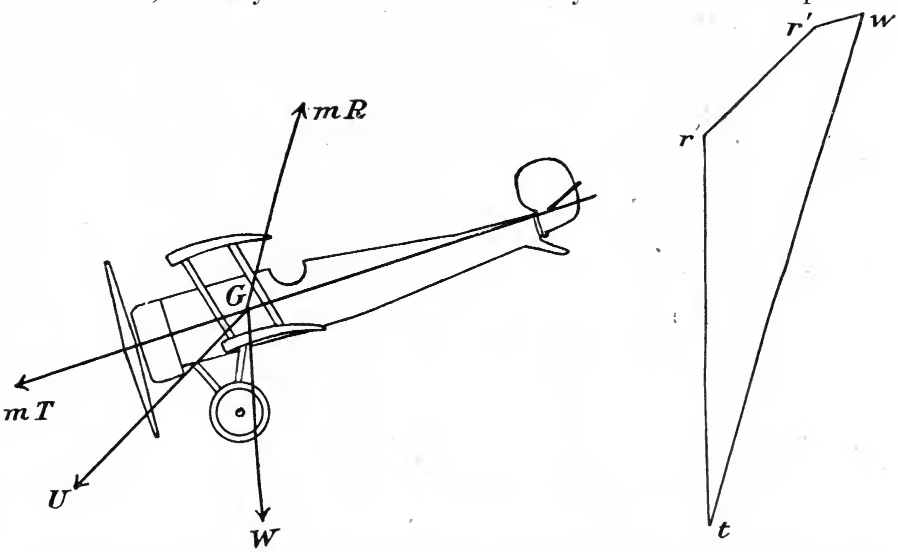

FIG. 40.—Impossibility of Uniform Rectilinear Acceleration of an Aeroplane in General.

and proportional to the weight W, tw parallel and proportional to the pressure mR, and wr' parallel and proportional to the thrust mT. Then rr' represents the resultant in magnitude and direction. For rectilinear accelerated or retarded motion rr' must be parallel to the fixed direction of motion. Thus the required propeller thrust depends on the air pressure, *i.e.* on the velocity, and a continual attendance to the engine is required. In other words, it is impossible, in general, to obtain rectilinear motion with variable velocity.

The argument breaks down in one case, and that is when mR is itself vertical and the propeller axis is vertical, *i.e.* in the vertical dive, with engine working. If the machine is falling vertically in such a way that there is no horizontal component of the air pressure, and no moment due to the air pressure, we have W, mT, mR all in one vertical straight line. Fig. 40 becomes Fig. 41, and we merely have a case of *some* vertical

resultant force acting on the machine for any vertical velocity. Of course, the position of the elevator, and, therefore, the angle of attack, must be correctly chosen so as to give what is technically known as no lift, *i.e.* no force perpendicular to the direction of motion, at the same time as there is no moment. This is only possible approximately.

In the vertical dive the machine behaves very much like a parachute. The air pressure is now a resistance, which we can assume proportional to the square of the velocity. The propeller thrust helps the gravity; it also depends on the velocity, given a fixed condition of the engine. Using the notation of Chapter II., § 42, we now have

$$m\frac{d^2y}{dt^2} = mg - K\left(\frac{dy}{dt}\right)^2 + mT \quad . \quad . \quad (109)$$

where y is measured vertically downwards, mg is the weight, $K(dy/dt)^2$ is the air resistance, and T is some function of the velocity representing the propeller thrust per unit mass. The function T is apparently not known exactly, but we can certainly say that T *diminishes as the velocity increases.* Let k be such a velocity that

$$mg = Kk^2 - mT(k) \quad . \quad . \quad . \quad (110)$$

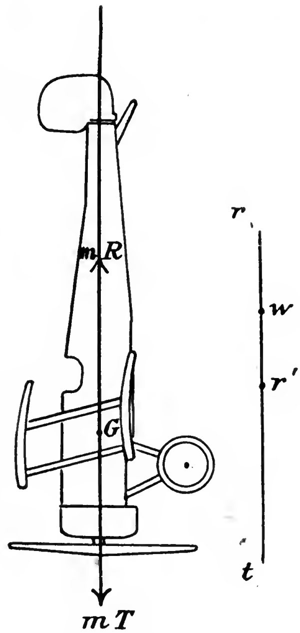

FIG. 41.—Vertical Dive.

If dy/dt is greater than k, then the quantity $K(dy/dt)^2 - mT$ is greater than mg, since the positive term increases as dy/dt increases, and the negative term decreases as dy/dt increases. Thus if the velocity is greater than k, the effect of the forces is to diminish it. But if dy/dt is less than k, the quantity $K(dy/dt)^2 - mT$ is less than mg, since the positive term decreases as dy/dt decreases, and the negative term increases as dy/dt decreases. Thus if the velocity is less than k, the effect of the forces is to increase it. So long as the velocity differs from k, the effect of the forces is to cause a continuous approach to k, *i.e.* the velocity tends towards the terminal velocity.

The interesting problem is when the thrust of the propeller does not exist. In this case the elevator position can be chosen so that there is a relative direction of motion, which gives only a drag and no lift, *i.e.* no force in a direction at right angles to this motion, whilst the air moment is zero. Hence, with no propeller thrust, we get a vertical dive for a certain elevator position, if the direction of motion is properly chosen. The motion is now exactly as in the parachute, since in (109) $T=0$. The terminal velocity is $\sqrt{mg/K}$.

90. **Method of Initial Motions.**—There is still another direction in which the solution of the problem of aeroplane motion can be prosecuted; this is by the method of initial motions.

In the equations of motion (81) let us suppose U_1, U_2, Ω_3 to have

values U_{10}, U_{20}, Ω_{30}, at time $t=0$, and suppose that for a sufficiently short interval of time it is permissible to write

$$\left.\begin{aligned} U_1 &= U_{10} + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \ldots, \\ U_2 &= U_{20} + \beta_1 t + \frac{\beta_2 t^2}{2!} + \frac{\beta_3 t^3}{3!} + \ldots, \\ \Omega_3 &= \Omega_{30} + \zeta_1 t + \frac{\zeta_2 t^2}{2!} + \frac{\zeta_3 t^3}{3!} + \ldots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

For Θ we clearly have

$$\Theta = \Theta_0 + \Omega_{30} t + \frac{\zeta_1 t^2}{2!} + \ldots$$

Now R_1, R_2, G_3, T depend on the motion of the aeroplane relatively to the axes fixed in itself, *i.e.* on U_1, U_2, Ω_3. Hence we have

$$R_1 = R_1(U_{10} + a_1 t + \ldots, \quad U_{20} + \beta_1 t \ldots, \quad \Omega_{30} + \zeta_1 t + \ldots)$$

$$= R_{10} + (a_1 t + \ldots)\frac{\partial R_{10}}{\partial U_{10}} + (\beta_1 t + \ldots)\frac{\partial R_{10}}{\partial U_{20}} + (\zeta_1 t + \ldots)\frac{\partial R_{10}}{\partial \Omega_{30}}$$

$$+ \tfrac{1}{2}\left\{(a_1 t + \ldots)^2 \frac{\partial^2 R_{10}}{\partial U_{10}^2} + \ldots + \ldots + 2(\beta_1 t + \ldots)(\zeta_1 t + \ldots)\frac{\partial^2 R_{10}}{\partial U_{20}\,\partial \Omega_{30}} + \ldots + \ldots\right\} + \ldots$$

where R_{10} is, in fact, the value $R_1(U_{10}, U_{20}, \Omega_{30})$. We see, then, that to the first power of t we have

$$R_1 - T = R_{10} - T_0 + U(a_x a_1 + b_x \beta_1 + f_x \zeta_1)t,$$

where U, a_x, b_x, f_x are the same as in the stability investigation. Similarly,

$$\begin{aligned} R_2 &= R_{20} + U(a_y a_1 + b_y \beta_1 + f_y \zeta_1)t, \\ G_3 &= G_{30} + U(a_3 a_1 + b_3 \beta_1 + f_3 \zeta_1)t. \end{aligned}$$

If we substitute in (81), we get, to the first power of t, the equations

$$\begin{aligned} & a_1 + a_2 t - U_{20}\Omega_{30} - U_{20}\zeta_1 t - \Omega_{30}\beta_1 t \\ & \qquad = g(\sin\Theta_0 + \Omega_{30} t \cos\Theta_0) + T_0 - R_{10} - U(a_x a_1 + b_x \beta_1 + f_x \zeta_1)t, \\ & \beta_1 + \beta_2 t + U_{10}\Omega_{30} + U_{10}\zeta_1 t + \Omega_{30} a_1 t \\ & \qquad = g(\cos\Theta_0 - \Omega_{30} t \sin\Theta_0) - \qquad R_{20} - U(a_y a_1 + b_y \beta_1 + f_y \zeta_1)t, \\ & \qquad \zeta_1 + \zeta_2 t = - G_{30} - U(a_3 a_1 + b_3 \beta_1 + f_3 \zeta_1)t. \end{aligned}$$

These give us the results

$$a_1 = g\sin\Theta_0 + T_0 - R_{10} + U_{20}\Omega_{30}, \quad \beta_1 = g\cos\Theta_0 - R_{20} - U_{10}\Omega_{30}, \quad \zeta_1 = -G_{30}, \quad . \quad (112)$$

and

$$\left.\begin{aligned} a_2 &= g\Omega_{30}\cos\Theta_0 - U(a_x a_1 + b_x \beta_1 + f_x \zeta_1) + U_{20}\zeta_1 + \Omega_{30}\beta_1, \\ \beta_2 &= -g\Omega_{30}\sin\Theta_0 - U(a_y a_1 + b_y \beta_1 + f_y \zeta_1) - U_{10}\zeta_1 - \Omega_{30} a_1, \\ \zeta_2 &= \qquad\qquad\qquad - U(a_3 a_1 + b_3 \beta_1 + f_3 \zeta_1). \end{aligned}\right\} \quad . \quad . \quad (113)$$

The coefficients a_1, β_1, ζ_1 are given directly by equations (112). When these are substituted in equations (113), we obtain a_2, β_2, ζ_2.

It is evident that this process can be continued indefinitely, *provided that certain information is available.* Returning to R_1, we see that, to the second power of t, we have

$$R_1 = R_{10} + \left(a_1 \frac{\partial R_{10}}{\partial U_{10}} + \beta_1 \frac{\partial R_{10}}{\partial U_{20}} + \zeta_1 \frac{\partial R_{10}}{\partial \Omega_{30}}\right) t$$

$$+ \tfrac{1}{2}\left(a_2 \frac{\partial R_{10}}{\partial U_{10}} + \beta_2 \frac{\partial R_{10}}{\partial U_{20}} + \zeta_2 \frac{\partial R_{10}}{\partial \Omega_{30}} + a_1{}^2 \frac{\partial^2 R_{10}}{\partial U_{10}{}^2} + \beta_1{}^2 \frac{\partial^2 R_{10}}{\partial U_{20}{}^2} + \zeta_1{}^2 \frac{\partial^2 R_{10}}{\partial \Omega_{30}{}^2}\right.$$

$$\left. + 2\beta_1\zeta_1 \frac{\partial^2 R_{10}}{\partial U_{20}\partial \Omega_{30}} + 2\zeta_1 a_1 \frac{\partial^2 R_{10}}{\partial \Omega_{30}\partial U_{10}} + 2a_1\beta_1 \frac{\partial^2 R_{10}}{\partial U_{10}\partial U_{20}}\right) t^2.$$

Hence it is necessary to know the values of the second differential coefficients of R_{10} with respect to the quantities U_{10}, U_{20}, Ω_{30}. Similar information is required for the second differential coefficients of R_{20}, G_{30}, T_0. If this information is known, the equations of motion (81), taken to the second power of t, now give us directly the values of a_3, β_3, ζ_3; but, as the student will readily verify, the expressions become very complicated. Proceeding in this way, we can, with sufficient labour, obtain any desired number of terms in the expansions for U_1, U_2, Ω_3, Θ; *i.e.* we obtain a statement of the motion which will hold for some time after $t = 0$. The more terms we take, the longer will be the range of time over which the motion thus found will be approximately true.

The disadvantages of this method are obvious enough. On the other hand, it possesses great advantages. Thus, even if we desire to find how an aeroplane settles down to steady motion from given initial conditions, it will be found that the method here given may be less laborious than the one in § 85, in which we have to find the values of the "arbitrary" constants A_1, A_2... from the initial conditions, and then expand complicated exponentials given by an algebraic equation of the fourth degree. The method is particularly useful when we have to deal with disturbances produced by gusts of wind (Chapter VIII.).

91. **Periodic Solutions.**—There is finally a process which should lead to useful results, based on the idea of periodicity. If the motion of an aeroplane is too far removed from the steady motion appropriate to its shape and condition of engine and elevator, and to find the initial part of the motion is not sufficient, we can attempt a solution on the idea that it is possible for the aeroplane to execute a series of motions of such a nature that the quantities U_1, U_2, Θ, and, therefore, also Ω_3, R_1, R_2, G_3, T, pass through sets of changes repeated periodically. We shall illustrate the idea by means of a very simple example.

Suppose that a body is moving in air sufficiently slowly to enable us to assume that the air pressures are proportional to the first power of displacements—not the second power, as is the case in aeroplane motions. In particular, let the body be a plane lamina, and let us use the axes of Fig. 42, the centre of gravity G being the origin, and GX along the lamina. Let the lamina be attached to a heavy body and the moment of inertia be so large that we may ignore the changes in the angular velocity, and also the effect of the angular motion on the air pressure. For a plane lamina we can assume the air pressure to be normal, and by the assumption of pressure proportional to displacements we put a pressure KU_2 per unit

mass along the Y axis, towards the centre of gravity G. The equations of motion are, therefore,

$$\left.\begin{aligned} \frac{dU_1}{dt} - U_2\frac{d\Theta}{dt} &= g\sin\Theta, \\ \frac{dU_2}{dt} + U_1\frac{d\Theta}{dt} &= g\cos\Theta - KU_2, \\ \frac{d^2\Theta}{dt^2} &= 0. \end{aligned}\right\} \qquad (114)$$

The third equation gives $d\Theta/dt$ = a constant which we shall call Ω. The first two equations are readily transformed into

$$\left.\begin{aligned} \frac{dU_1}{d\Theta} - U_2 &= \frac{g}{\Omega}\sin\Theta, \\ U_1 + \frac{dU_2}{d\Theta} + \frac{K}{\Omega}U_2 &= \frac{g}{\Omega}\cos\Theta. \end{aligned}\right\} \qquad (115)$$

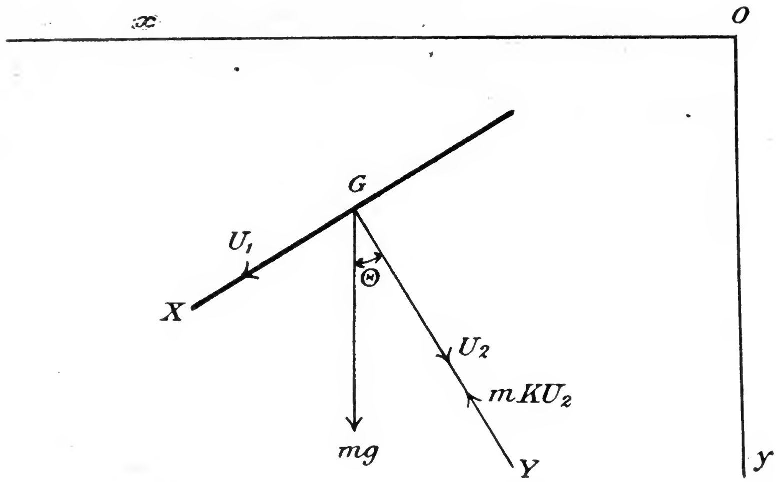

FIG. 42.—Lamina : Simplified Law of Air Resistance.

We can easily eliminate U_2, and we get

$$\frac{d^2U_1}{d\Theta^2} + \frac{K}{\Omega}\frac{dU_1}{d\Theta} + U_1 = \frac{2g}{\Omega}\cos\Theta + \frac{Kg}{\Omega^2}\sin\Theta \qquad (116)$$

Assuming K to be less than 2Ω, we have immediately

$$U_1 = A_1 e^{-\frac{K}{2\Omega}\Theta}\sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\left(\Theta + B_1\right) - \frac{g}{\Omega}\cos\Theta + \frac{2g}{K}\sin\Theta \qquad (117)$$

Substituting in the first equation (115), we get

$$\begin{aligned} U_2 = &-\frac{K}{2\Omega}A_1 e^{-\frac{K}{2\Omega}\Theta}\sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\left(\Theta + B_1\right) \\ &+ \left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}} A_1 e^{-\frac{K}{2\Omega}\Theta}\cos\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\left(\Theta + B_1\right) + \frac{2g}{K}\cos\Theta \qquad (118) \end{aligned}$$

A_1, B_1 are arbitrary constants depending on the initial conditions.

Similar results can be readily obtained for $K = 2\Omega$ and $K > 2\Omega$.

After a certain time, when the angle Θ has increased a considerable amount, the exponential $e^{-\frac{K}{2\Omega}\Theta}$ in (117, 8) will be so small that it can be put zero. The motion is given by

$$\left.\begin{aligned} U_1 &= \frac{2g}{K}\sin\Theta - \frac{g}{\Omega}\cos\Theta, \\ U_2 &= \frac{2g}{K}\cos\Theta. \end{aligned}\right\} \quad \ldots \ldots \quad (119)$$

It will be found that (119) holds for all values of K and Ω. The angle Θ can be put $\Omega t + \Theta_0$ if we choose, Θ_0 being the initial value of Θ. But let us examine the path. The horizontal and downward vertical velocities are

$$\left.\begin{aligned} \frac{dx}{dt} &= U_1\cos\Theta - U_2\sin\Theta = -\frac{g}{2\Omega} - \frac{g}{2\Omega}\cos 2\Theta, \\ \frac{dy}{dt} &= U_1\sin\Theta + U_2\cos\Theta = \frac{2g}{K} - \frac{g}{2\Omega}\sin 2\Theta. \end{aligned}\right\} \quad \ldots \quad (120)$$

It is thus clear that the direction of motion is, on the whole, along a line making an angle $\tan^{-1}(K/4\Omega)$ on the same side of the downward vertical as the sense of the rotation. But the motion is really periodic. The horizontal and downward vertical co-ordinates at any time t are

$$\begin{aligned} x = \int_0^t \frac{dx}{dt}\,dt &= -\frac{g}{2\Omega}t - \frac{g}{2\Omega}\int_0^t \cos 2(\Omega t + \Theta_0)\,dt \\ &= -\frac{g}{2\Omega}t - \frac{g}{4\Omega^2}\{\sin 2(\Omega t + \Theta_0) - \sin 2\Theta_0\} \\ &= -\frac{g}{2\Omega}t - \frac{g}{2\Omega^2}\sin\Omega t\cos(\Omega t + 2\Theta_0), \end{aligned}$$

$$\begin{aligned} y = \int_0^t \frac{dy}{dt}\,dt &= \frac{2g}{K}t - \frac{g}{2\Omega}\int_0^t \sin 2(\Omega t + \Theta_0)\,dt \\ &= \frac{2g}{K}t + \frac{g}{4\Omega^2}\{\cos 2(\Omega t + \Theta_0) - \cos 2\Theta_0\} \\ &= \frac{2g}{K}t - \frac{g}{2\Omega^2}\sin\Omega t\sin(\Omega t + 2\Theta_0). \end{aligned}$$

If we measure t from the time when Θ was zero, we can put $\Theta_0 = 0$. The co-ordinates of the centre of gravity of the lamina are

$$\left.\begin{aligned} x &= -\frac{g}{2\Omega}t - \frac{g}{2\Omega^2}\sin\Omega t\cos\Omega t, \\ y &= \frac{2g}{K}t - \frac{g}{2\Omega^2}\sin^2\Omega t, \end{aligned}\right\} \quad \ldots \ldots \quad (121)$$

whilst the direction of the lamina is given by $\Theta = \Omega t$. In Fig. 43 the path of the centre of gravity and the direction of the lamina (indicated by an arrow) are plotted for several moderate ratios of K/Ω and a fixed value of K. The lamina "loops the loop" indefinitely often; the motion

for half a complete turn is shown. If K/Ω is large, the path may actually be upwards at some part of it, as in Fig. 44.

92. **Terminal Motion; Rate of Fall.**—We have, in fact, an extended definition of "terminal motion." We have seen that a particle tends towards a terminal motion, the same being true of a parachute and of an aeroplane diving vertically. We now see that we can define terminal motion to be a state of periodic motion to which a body tends to approxi-

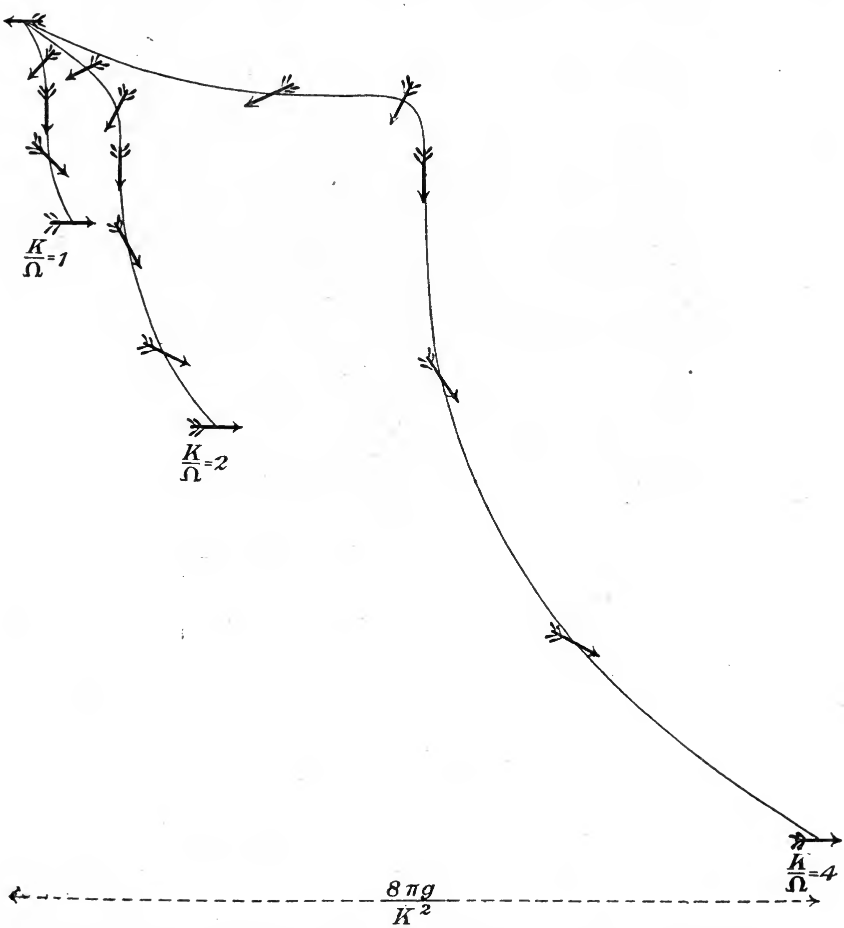

FIG. 43.—Lamina with Simplified Law of Air Resistance. Terminal Motion for Various Values of K/Ω.

mate under the influence of gravity and the air pressures. We cannot define a terminal velocity, but we can say that the terminal velocity components for a direction Θ with the horizontal are those given by equations (119).

A few remarkable properties of the motion just considered throw some light on resisted motions in general. It has been seen that the centre of gravity describes a path whose average is a straight line inclined to the vertical. The angle with the vertical is $\tan^{-1}(K/4\Omega)$. We see

that the smaller Ω, the greater is this angle. At first sight this may appear to be obviously wrong, since if the plane lamina were initially horizontal and Ω were zero, we should have a vertical fall. The answer is that we must not push such a result to its logical extreme. We assumed constant angular velocity, basing it on the fact that the moments due to air pressure are negligible and the moment of inertia is great. We must then avoid using zero value of Ω, which cannot certainly be maintained. The greatness of the angle for small Ω is to be looked for in the numerator K rather than in the denominator Ω. The deviation of the path of the centre of gravity from the vertical is due to the effect of the air, and this is greater, the greater is K/Ω.

Another interesting result is that the average rate of vertical fall is $2g/K$, and, therefore, quite independent of the angular velocity, *so long as*

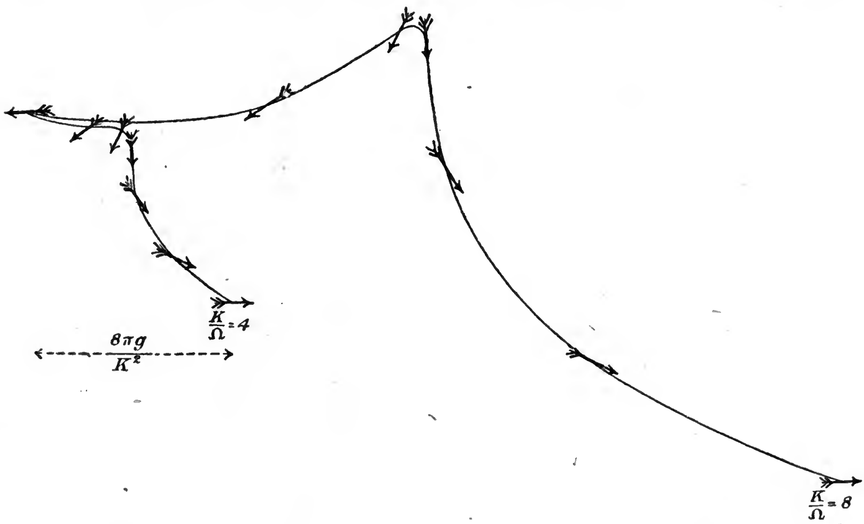

FIG. 44.—Lamina with Simplified Law of Air Resistance. Terminal Motion for Large Value of K/Ω.

an angular velocity exists. If the body could fall as a parachute, the terminal velocity would be g/K, or just half as great.

The way in which the terminal motion is approximated to with various initial conditions is indicated in Figs. 45, 46. In each graph the beginning is at the top end; at the lower end the motion is practically indistinguishable from the terminal motion. The number with the arrow at the beginning of each path shows the initial velocity of the centre of gravity; the arrow gives the direction of the initial velocity, and the number gives the ratio of this velocity to the average rate of vertical fall.

Now it is clear that we can have similar motion in the case of *any* body falling under gravity in a fluid like air. The analytical statement of the motion would not, of course, be so simple, but the general principle must hold; if at a given orientation of the body it has certain translational and rotational motions, which we call the terminal velocities for this orientation, then the motion will be periodic and will continue repeating itself indefinitely. The general nature of the motion will be a fall along

a line inclined to the vertical, whilst the body will turn over and over or oscillate.

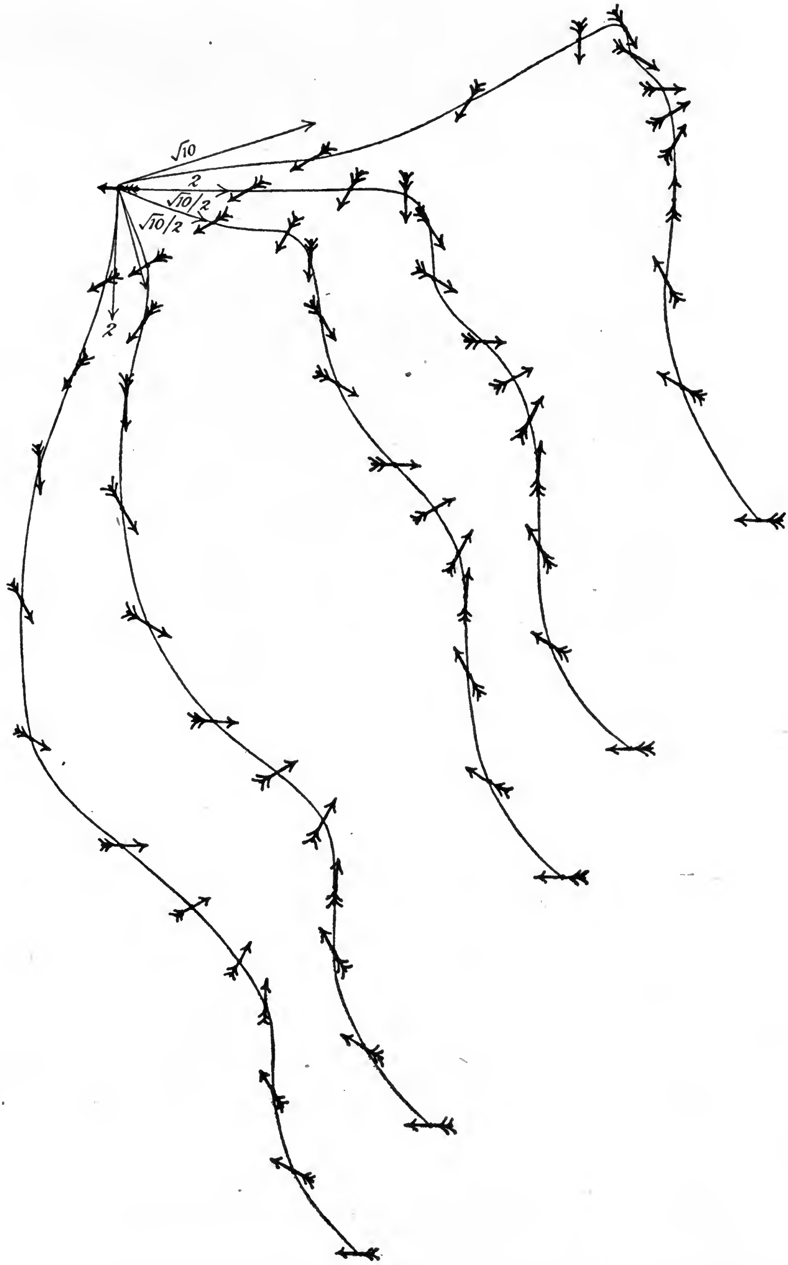

Fig. 45.—Lamina with Simplified Law of Air Resistance. Motion with various Initial Velocities, Lamina Horizontal Initially.

The mathematical theory of looping motion of an aeroplane with no propeller thrust is thus reduced to a discussion of the periodic solutions

of the equations of motion (81). We are not yet in a position to do any useful work with this method, because the forms of the functions R_1, R_2, G_3 in terms of U_1, U_2, $d\Theta/dt$ must first be ascertained. But, given this information, even if only approximate and based on experiment, it should be possible to obtain results in the manner here indicated.

For the case of a lamina such as described in § 91, but with laws of

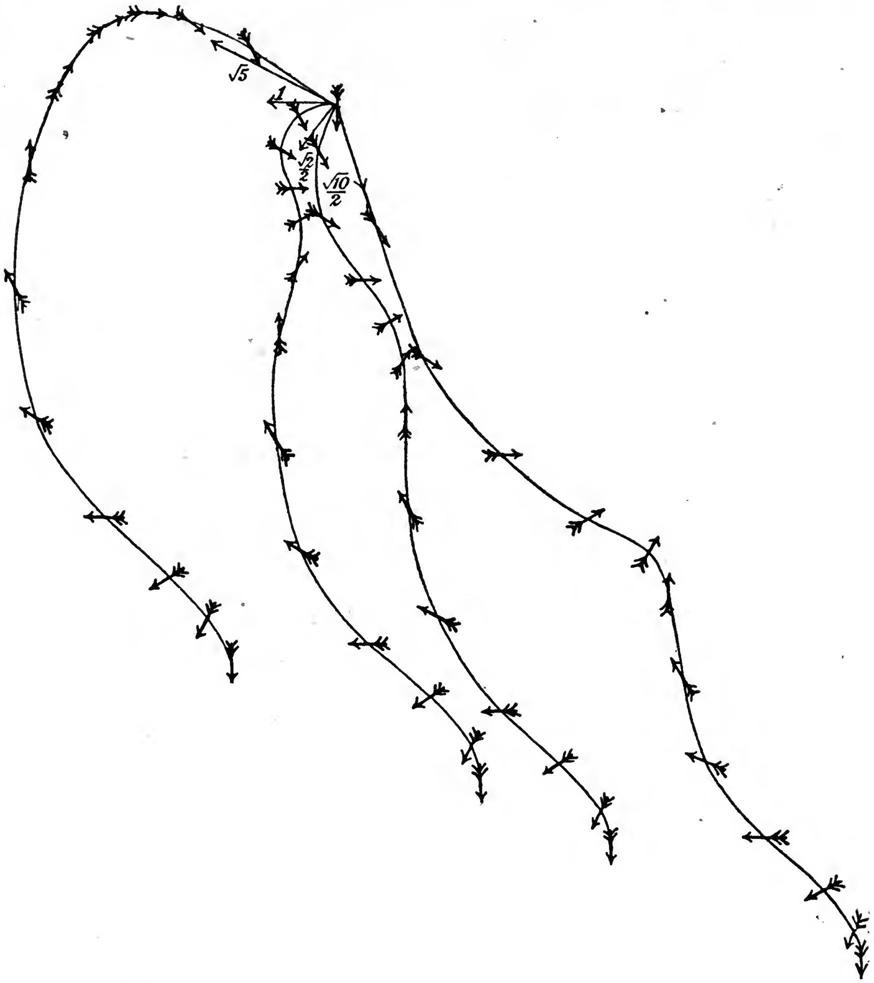

Fig. 46.—Lamina with Simplified Law of Air Resistance. Motion with various Initial Velocities, Lamina Vertical Initially.

air pressure more in agreement with facts, the results indicated have been obtained by the author, using graphical methods of integration (see *Proc. Roy. Soc.* A, 95, 1919, pp. 516–32).

93. **The Kite; The General Equations of Motion in Two Dimensions; Single String, Light and Inextensible.**—An interesting and important problem in connection with our subject is that of the kite. A kite can be considered as an aerofoil kept in statical

equilibrium by its weight, the air pressures due to relative wind, and the tension of a string. We have seen that it is impossible for a body to be supported in the air by wind pressure alone (except in the very special case when the direction and strength of the wind are such as to produce an air pressure exactly equal and opposite to the weight). The introduction of a third force, the tension of the string, makes equilibrium possible.

In the aeroplane problem we have taken the air to be "at rest" and the aeroplane to move through it. In the kite problem we shall take the kite to be at rest and the air to move past it in the form of a steady wind, except in so far as disturbances in the air motion are introduced by the very presence of the kite. We shall suppose the string to be light and

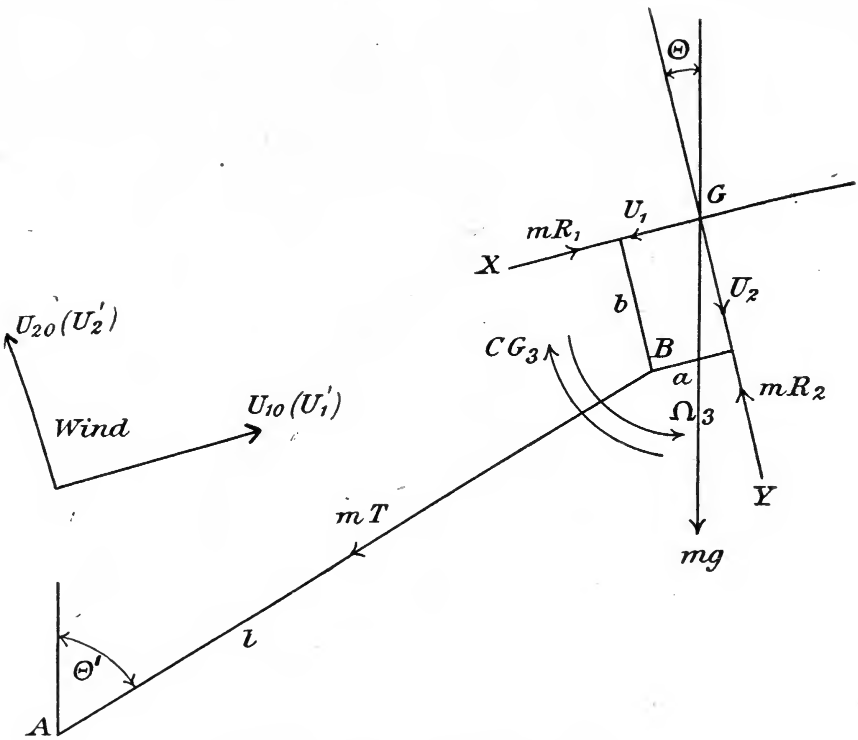

Fig. 47.—The Kite: Two-Dimensional Problem.

inextensible, and also, at present, we shall take the kite to have a plane of symmetry which contains the string and the direction of the wind, and also any motions that take place.

In Fig. 47, which represents a section by the plane of symmetry, let A be the point where the string is held, B the end of the string attached to a point of the kite, G the centre of gravity, and GX, GY axes fixed in the kite. Let (a, b) be the co-ordinates of B relatively to these axes, $AB = l$, and let Θ be the angle between GX and the horizontal, Θ' the angle between AB and the upward vertical. If mg is the weight, $C = mk_3^2$ the moment of inertia of the kite about an axis through G perpendicular to the plane of the figure, mT the tension of the string, U_1, U_2 the X, Y velocity components of G, Ω_3 the angular velocity of the kite from X to Y,

and mR_1, mR_2, $mk_3^2G_3$ the air forces backwards along GX, GY and couple from Y to X, then we have the following equations of motion:

$$\left.\begin{aligned} \frac{dU_1}{dt} - U_2\Omega_3 &= g\sin\Theta + T\sin(\Theta' + \Theta) - R_1, \\ \frac{dU_2}{dt} + U_1\Omega_3 &= g\cos\Theta + T\cos(\Theta' + \Theta) - R_2, \\ \frac{d\Omega_3}{dt} &= \frac{T}{k_3^2}\{a\cos(\Theta' + \Theta) - b\sin(\Theta' + \Theta)\} - G_3, \end{aligned}\right\} \quad . \quad . \quad (122)$$

the moment of the tension about G being a into the Y component *minus* b into the X component. But we have *five* unknown quantities: U_1, U_2, Θ, Θ', T, ($\Omega_3 \equiv d\Theta/dt$, and is, therefore, determined by Θ) whereas we only have *three* equations in (122). We, therefore, need *two* more equations. These are supplied by the following two geometrical facts:

(i) The velocity component of B along the string is zero;

(ii) The velocity component of B perpendicular to the string is $l\dfrac{d\Theta'}{dt}$ in the direction of increasing Θ'. But the velocity components of B along the directions X, Y are $U_1 - b\Omega_3$, $U_2 + a\Omega_3$. Hence we have

$$\left.\begin{aligned} (U_1 - b\Omega_3)\sin(\Theta' + \Theta) + (U_2 + a\Omega_3)\cos(\Theta' + \Theta) &= 0, \\ -(U_1 - b\Omega_3)\cos(\Theta' + \Theta) + (U_2 + a\Omega_3)\sin(\Theta' + \Theta) &= l\frac{d\Theta'}{dt}. \end{aligned}\right\} \quad (123)$$

We now have *five* equations to determine the *five* unknown quantities, and the analytical formulation of the problem is established.

94. **Equilibrium.**—Consider first the equilibrium position, in which we can put

$$U_1 = U_2 = \Omega_3 = 0,\ \ \Theta = \Theta_0,\ \ \Theta' = \Theta'_0,\ \ R_1 = R_{10},\ \ R_2 = R_{20},\ \ G_3 = G_{30},\ \ T = T_0,$$

the zero suffixes denoting the constant values in equilibrium. The geometrical conditions (123) now have no significance, and the equations (122) become

$$\left.\begin{aligned} R_{10} &= g\sin\Theta_0 + T_0\sin(\Theta'_0 + \Theta_0), \\ R_{20} &= g\cos\Theta_0 + T_0\cos(\Theta'_0 + \Theta_0), \\ G_{30}k_3^2 &= T_0\{a\cos(\Theta'_0 + \Theta_0) - b\sin(\Theta'_0 + \Theta_0)\}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (124)$$

The quantities R_1, R_2, G_3 in the general problem depend on the shape of the kite, the air density, and the steady motion of the air past the kite. Taking the kite itself and the air density as given, then R_1, R_2, G_3 depend on the component relative velocities and angular velocity of the kite, namely, $U_1' + U_1$, $U_2' + U_2$, Ω_3, where U_1', U_2' are the wind components along the X, Y axes towards G, as in Fig. 47. Hence, if in equilibrium we write U_{10} for U_1', U_{20} for U_2', to emphasise the analogy with the aeroplane problem, we get $R_{10} \equiv R_{10}\ (U_{10},\ U_{20},\ 0)$, with similar expressions for R_{20} and G_{30}. Since the wind is supposed known, we see that U_{10}, U_{20}, and, therefore, also R_{10}, R_{20}, G_{30}, are merely functions of Θ_0. The equilibrium conditions (124) give *three* equations for the *three* quantities Θ_0, Θ'_0, T_0, which can thus be evaluated. Thus for any given kite, air density, and wind we have a definite equilibrium position (*which is independent of the length of the string*).

The analogy with the aeroplane steady motion is obvious. The

tension of the string takes the place of the propeller thrust, and the equilibrium takes the place of the steady motion.

95. **Longitudinal Stability.**—Judging by the experience gained in the case of the aeroplane, we do not attempt anything more than the problem of stability. If the kite is slightly disturbed from its position of equilibrium, or if for any reason the actual position of the kite (whether it is at rest or not) is not that which is appropriate to the given kite and wind, will the kite tend to assume the correct position of equilibrium? Or will it deviate more and more from the equilibrium position and perhaps come to grief? The method of investigation is essentially the same as that already described for the aeroplane.

In the motion resulting from the disturbance let us put

$$U_1 = u_1, \quad U_2 = u_2, \quad \Omega_3 = \omega_3, \quad \Theta = \Theta_0 + \theta, \quad \Theta' = \Theta'_0 + \theta',$$

where u_1, u_2, ω_3, θ, θ' are to be assumed small enough to justify our neglecting their squares and products, etc.; and let us discover whether we thus get equations consistent with the assumption of small deviation from equilibrium. We take the wind components U_{10}, U_{20} to refer to the directions of X, Y in the equilibrium position. The student will readily verify that

$$\begin{aligned}R_1 &\equiv R_1(U_{10} + u_1 + U_{20}\,\theta, \;\; U_{20} + u_2 - U_{10}\,\theta, \;\; \omega_3) \\ &\equiv R_{10}(U_{10}, \;\; U_{20}, \;\; 0) + (u_1 + U_{20}\,\theta)\frac{\partial R_{10}}{\partial U_{10}} + (u_2 - U_{10}\,\theta)\frac{\partial R_{10}}{\partial U_{20}} + \omega_3\frac{\partial R_{10}}{\partial \Omega_{30}} \\ &= R_{10} + U(a_x\,.\,\overline{u_1 + U_{20}\,\theta} + b_x\,.\,\overline{u_2 - U_{10}\,\theta} + f_x\,\omega_3),\end{aligned}$$

where $U = (U_{10}^2 + U_{20}^2)^{\frac{1}{2}}$, and a_x, b_x, f_x have meanings similar to those assigned in the aeroplane theory. Similarly,

$$\begin{aligned}R_2 &= R_{20} + U(a_y\,.\,\overline{u_1 + U_{20}\,.\,\theta} + b_y\,.\,\overline{u_2 - U_{10}\,.\,\theta} + f_y\,\omega_3), \\ G_3 &= G_{30} + U(a_3\,.\,\overline{u_1 + U_{20}\,.\,\theta} + b_3\,.\,\overline{u_2 - U_{10}\,.\,\theta} + f_3\,\omega_3).\end{aligned}$$

Putting $T = T_0 + \delta T$ and substituting in (122), using (124), we deduce

$$\left.\begin{aligned}\frac{du_1}{dt} &= g\cos\Theta_0\,.\,\theta + \delta T\sin(\Theta'_0 + \Theta_0) + T_0\cos(\Theta'_0 + \Theta_0)(\theta' + \theta) \\ &\qquad - U(a_x\,.\,\overline{u_1 + U_{20}\,.\,\theta} + b_x\,.\,\overline{u_2 - U_{10}\,.\,\theta} + f_x\,\omega_3), \\ \frac{du_2}{dt} &= -g\sin\Theta_0\,.\,\theta + \delta T\cos(\Theta'_0 + \Theta_0) - T_0\sin(\Theta'_0 + \Theta_0)(\theta' + \theta) \\ &\qquad - U(a_y\,.\,\overline{u_1 + U_{20}\,.\,\theta} + b_y\,.\,\overline{u_2 - U_{10}\,.\,\theta} + f_y\,\omega_3), \\ \frac{d^2\theta}{dt^2} &= \frac{\delta T}{k_3^2}\{a\cos(\Theta'_0 + \Theta_0) - b\sin(\Theta'_0 + \Theta_0)\} \\ &\qquad - \frac{T_0}{k_3^2}\{a\sin(\Theta'_0 + \Theta_0) + b\cos(\Theta'_0 + \Theta_0)\}(\theta' + \theta) \\ &\qquad - U(a_3\,.\,\overline{u_1 + U_{20}\,.\,\theta} + b_3\,.\,\overline{u_2 - U_{10}\,.\,\theta} + f_3\,\omega_3),\end{aligned}\right\} \quad (125)$$

whilst the geometrical conditions (123) become

$$\left.\begin{aligned}&(u_1 - b\omega_3)\sin(\Theta'_0 + \Theta_0) + (u_2 + a\omega_3)\cos(\Theta'_0 + \Theta_0) = 0, \\ -&(u_1 - b\omega_3)\cos(\Theta'_0 + \Theta_0) + (u_2 + a\omega_3)\sin(\Theta'_0 + \Theta_0) = l\frac{d\theta'}{dt} = l\frac{d}{dt}(\theta' + \theta) - l\frac{d\theta}{dt}.\end{aligned}\right\} \quad (126)$$

The quantities Θ_0, Θ'_0, T_0, and the air pressure derivatives can be supposed to be known constants. We, therefore, have five linear differential equations with constant coefficients for the five quantities u_1, u_2, θ, $\theta' + \theta$, δT.

If we follow out the method given for the aeroplane, we get for the conditions of stability that the real parts of the roots of the following determinantal equation in λ shall all be negative:

$$\begin{vmatrix} \lambda+a_x, & b_x, & f_x\lambda+\frac{U_{20}}{U}a_x-\frac{U_{10}}{U}b_x-\frac{g}{U^2}\cos\Theta_0, & \cos a, & \sin a \\ a_y, & \lambda+b_y, & f_y\lambda+\frac{U_{20}}{U}a_y-\frac{U_{10}}{U}b_y+\frac{g}{U^2}\sin\Theta_0, & -\sin a, & \cos a \\ a_3, & b_3, & \lambda^2+f_3\lambda+\frac{U_{20}}{U}a_3-\frac{U_{10}}{U}b_3, & -\frac{a\sin a+b\cos a}{k_3^2}, & \frac{a\cos a-b\sin a}{k_3^2} \\ \cos a, & -\sin a, & -(a\sin a+b\cos a+l)\lambda, & -\frac{lU^2}{T_0}\lambda, & 0 \\ \sin a, & \cos a, & (a\cos a-b\sin a)\lambda, & 0, & 0 \end{vmatrix}=0, \quad (127)$$

where we have written $a \equiv \Theta'_0 + \Theta_0$ for brevity.

We have not yet chosen the body axes GX, GY, which we are at liberty to take in any convenient positions. Let us, therefore, by analogy with the aeroplane mathematics, make the X axis parallel to the direction of the string when the kite is in equilibrium, the wind components being still denoted by U_{10}, U_{20} parallel to the axes thus chosen. We get $\Theta'_0+\Theta_0=90°$, so that $\sin a=1$, $\cos a=0$. Equation (127) simplifies to

$$\begin{vmatrix} \lambda+a_x, & b_x, & f_x\lambda+\frac{U_{20}}{U}a_x-\frac{U_{10}}{U}b_x-\frac{g}{U^2}\cos\Theta_0, & 0, & 1 \\ a_y, & \lambda+b_y, & f_y\lambda+\frac{U_{20}}{U}a_y-\frac{U_{10}}{U}b_y+\frac{g}{U^2}\sin\Theta_0, & -1, & 0 \\ a_3, & b_3, & \lambda^2+f_3\lambda+\frac{U_{20}}{U}a_3-\frac{U_{10}}{U}b_3, & -\frac{a}{k_3^2}, & -\frac{b}{k_3^2} \\ 0, & -1, & -(l+a)\lambda, & -\frac{lU^2}{T_0}\lambda, & 0 \\ 1, & 0, & -b\lambda, & 0, & 0 \end{vmatrix}=0. \quad (128)$$

Further simplifications are left to the student as exercises in the manipulation of the rows and columns of determinants. It will be seen that we have again a *biquadratic* equation for λ. The conditions of stability are, therefore, that the coefficients of this biquadratic and their Routh's discriminant shall all have the same sign.

96. Various cases can be investigated in this way. Thus the string may be so long that the change in Θ' can be omitted from consideration. If we leave out Θ' in the equations (125) and discard the second geometrical condition (126), which now has no meaning, we get a determinantal equation for λ of the third degree. The conditions of stability are those given in § 82, in terms of the coefficients of this cubic equation.

There is the possibility that the string is extensible. We discard the first geometrical condition (123) and replace it by an equation for the tension in terms of the modulus of extension of the string, whilst in the second condition we use for l the extended length of the string. There is now an additional variable, namely, the length of the string, and the result is correspondingly complicated.

The kite may also in its statical equilibrium be propelled at a constant rate in some fixed direction, either by means of the point A being forced to move uniformly, or by a uniform increase in the length of the string, or both. We merely add on to the velocity of the wind a velocity equal and opposite to the rate at which the kite is propelled and the string is served out.

97. **Forked String.**—An important practical problem in connection with the kite is when the string is not attached directly to a point in the kite, but is forked as in Fig. 48, where the string AB is joined to the kite by means of strings BC, BD. The motion of the kite is now defined by the motion of the point of bifurcation, B, and the motion round this point. It is, therefore, convenient to make B the origin of co-ordinates, the triangle BCD being invariable in size and shape, so that the axes

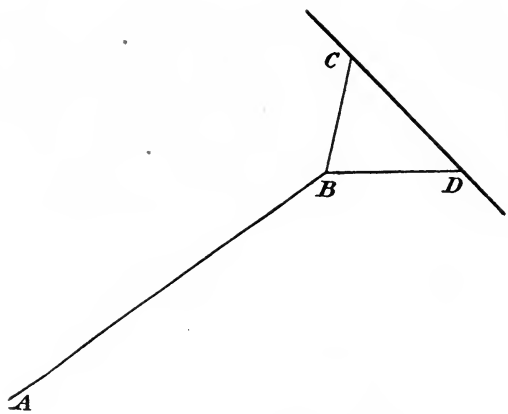

FIG. 48.—The Kite: Forked String.

through B can be made fixed in the kite. Convenient axes are the prolongation of AB and the line perpendicular to it, as suggested in § 95 for the form (128) of the equation for λ. The tensions of the strings BC, BD can be replaced absolutely by that of AB. This is because the tension in BC, BD must be equivalent to that in AB, as there is no mass at B, and the forces meeting at B must consequently balance.

There is no modification in the equation of angular acceleration, if care is taken to write down correctly the moment of the tension about the centre of gravity. In constructing the equations for the linear accelerations of the centre of gravity, we must write down the changes of the velocities of the centre of gravity *in space*, using the method of moving axes (§ 74).

Another case is when CBD is one string which passes through a loop at B.

A full treatment of the kite problem by Bryan is given in the *Aeronautical Journal*, October, 1915.

APPENDIX TO CHAPTER III

In order to follow many of the investigations published during the past ten years, particularly those contained in the *Technical Reports of the Advisory Committee for Aeronautics*, the student will have to reconstruct the mathematics in terms of the notation there adopted. The pedagogical discipline of this reconstruction is in itself of great value, and in what follows we shall merely indicate this notation and the results: the derivation of the results is left to the student for exercise. We consider here the longitudinal problem.

Fig. 33A shows the alternative notation. The space or earth axes Ox, Oz are taken positive to the right and upwards respectively. The

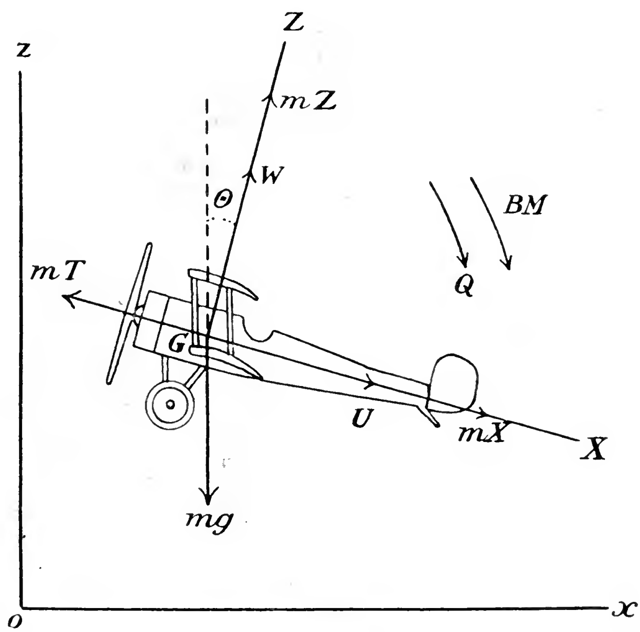

FIG. 33A.—Longitudinal Motion: Alternative Notation.

body axes GX, GZ are such that when G is at O and GX along Ox, then GZ coincides with Oz. Actually GX is the direction of the relative wind in normal flight, so that it points backwards as seen by the pilot; GZ is perpendicular to GX and points upwards as seen by the pilot.

The velocity components are U, W along the directions of XZ, whilst the rotation is represented by an angular velocity Q from Z to X. The orientation of the body axes relatively to the space axes is indicated by the angle Θ between the axis of X and the axis of x, measured positive from Z to X. Since the body axes now represent the general direction of the relative wind, we represent the effect of the air-resistance by the following symbols: mX is the component of air-resistance along the X

axis, mZ is the component along the Z axis, and the couple due to air-resistance is represented by BM: m is the mass of the machine and B is the moment of inertia about an axis through G perpendicular to the plane of motion.

We now get the following equations of motion :—

$$\frac{dU}{dt} + QW = g \sin \Theta + X - T,$$

$$\frac{dW}{dt} - QU = - g \cos \Theta + Z,$$

$$\frac{dQ}{dt} = M,$$

so that in steady motion we have

$$g \sin \Theta_0 + X_0 = T_0, \quad - g \cos \Theta_0 + Z_0 = 0, \quad M_0 = 0,$$

where X_0, Z_0, M_0 are functions of U_0, W_0, the velocity components in steady motion ; $Q_0 = 0$ of course ; T_0 is the thrust per unit mass in the steady motion.

If now there is a disturbance of the steady motion, we write

$$\begin{aligned} U = U_0 + u, \quad W = W_0 + w, \quad Q = q, \quad \Theta &= \Theta_0 + \theta, \\ X &= X_0 + uX_u + wX_w + qX_q, \\ Z &= Z_0 + uZ_u + wZ_w + qZ_q, \\ M &= \qquad uM_u + wM_w + qM_q, \end{aligned}$$

where we suppose the derivatives due to T are absorbed in those for X, Z, M.

The equations of motion, assuming that u, w, θ (and therefore also q) remain small and that we only retain first powers of these quantities, are

$$\frac{du}{dt} + W_0 \frac{d\theta}{dt} = g\theta \cos \Theta_0 + uX_u + wX_w + qX_q,$$

$$\frac{dw}{dt} - U_0 \frac{d\theta}{dt} = g\theta \sin \Theta_0 + uZ_u + wZ_w + qZ_q,$$

$$\frac{d^2\theta}{dt^2} = \qquad uM_u + wM_w + qM_q.$$

If we express u, w, θ as sums of exponentials like $e^{\lambda t}$, we get for λ the determinantal equation

$$\begin{vmatrix} \lambda - X_u, & - X_w, & \overline{W_0 - X_q} \,.\, \lambda - g \cos \Theta_0 \\ - Z_u, & \lambda - Z_w, & - \overline{U_0 + Z_q} \,.\, \lambda - g \sin \Theta_0 \\ - M_u, & - M_w, & \lambda^2 - M_q \,.\, \lambda \end{vmatrix} = 0.$$

This is again a biquadratic equation in λ, and we write it

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

and the conditions of stability are once more that

$$A_1,\ B_1,\ C_1,\ D_1,\ E_1, \text{ and } H_1 \equiv B_1C_1D_1 - A_1D_1^2 - E_1B_1^2$$

all have the same sign.

In a further development of this notation, adopted by the Technical Terms Committee of the Royal Aeronautical Society, the axes x, z are in directions opposite to those of x, z just defined, and similarly the directions of the X, Z axes are opposite in direction to the X, Z axes just defined.

EXERCISES (Chapter III)

1. Discuss the graphical statics of steady motion when the centre of gravity of the machine is not in the propeller axis.

2. For given state of the engine it can be assumed that the propeller thrust diminishes as the velocity increases, becoming zero for a certain velocity and then becoming negative. On the other hand, the air-resistance increases as the velocity increases. Hence deduce that in order to descend in steady flight in a given direction in space, at a moderate inclination to the horizontal, two possible motions exist, one with large velocity and small angle of attack, the other with small velocity and larger angle of attack.

3. Show that in order to fly steadily in a horizontal direction there is a range of possible velocities, assuming a limit to the possible power of the engine.

4. Assuming that air-resistances are proportional to the density of the air, examine the effect of change of air density on the possibility of flight in a horizontal direction: it must be remembered that the propeller thrust is also dependent on air-resistances.

5. Show that a level exists above which the density of the air is too small to admit of horizontal flight: this is called the *ceiling*.

6. Investigate the motion in a vertical dive with propeller thrust, taking the thrust to decrease linearly as the velocity increases. Let U_0 be the velocity for zero thrust and let

$$mT = mA(U_0 - U).$$

Find the terminal velocity. Is it greater or less than when the propeller thrust is just zero as suggested in the text?

7. Assuming the derivatives a, b, f to be fairly constant for steady motions not far from normal motion, with U_{20} zero, discuss how climbing affects the stability, and shew that there is some danger of instability through the value of D_1 becoming negative (b_3 is negative, Ch. VII).

8. In the problem of § 91 take $K = 2\Omega$, and plot the path of the centre of gravity for various initial conditions :—

(i) Lamina horizontal initially and let fall from rest;
(ii) Lamina vertical initially and let fall from rest;
(iii) Lamina horizontal initially and projected vertically upwards;
(iv) Lamina horizontal initially and projected vertically downwards;
(v) Lamina vertical initially and projected vertically upwards.

In each case the lamina is to have the constant spin Ω.

9. Assuming that the air-resistance for any element of area of a lamina is normal and varies as the normal velocity, find the couple due to air-resistance for a symmetrical lamina moving in two dimensions. Show that the motion of the centre of gravity can be obtained in terms of Bessel functions.

10. Examine the vertical oscillation of a spherical balloon about its position of equilibrium, taking the air-resistance to be proportional to the square of the velocity. Neglect the effect of variations in the air-density.

11. Discuss the motion of the lamina in Question 9 assuming it to be falling vertically like a parachute with its terminal velocity and to be slightly disturbed. Show that the oscillation must lead to instability.

12. If the lamina in Question 11 has a weight rigidly attached to it so that the centre of gravity of the whole is below the lamina, find the conditions for stability.

CHAPTER IV

THREE-DIMENSIONAL MOTION OF THE AEROPLANE: LATERAL STABILITY: CIRCLING AND HELICAL FLIGHT: THE KITE

98. We now come to the three-dimensional motion of an aeroplane. Having learnt by experience that we must limit the generality of the problem in order to make possible any attempts at solution, we shall say at once that we consider the air resistances (including the propeller thrust) to be given by the instantaneous motion of the machine relatively to the air "at rest," on the assumption that the motions set up in the air are steady motions. Further, we shall ignore variations in the density of the air.

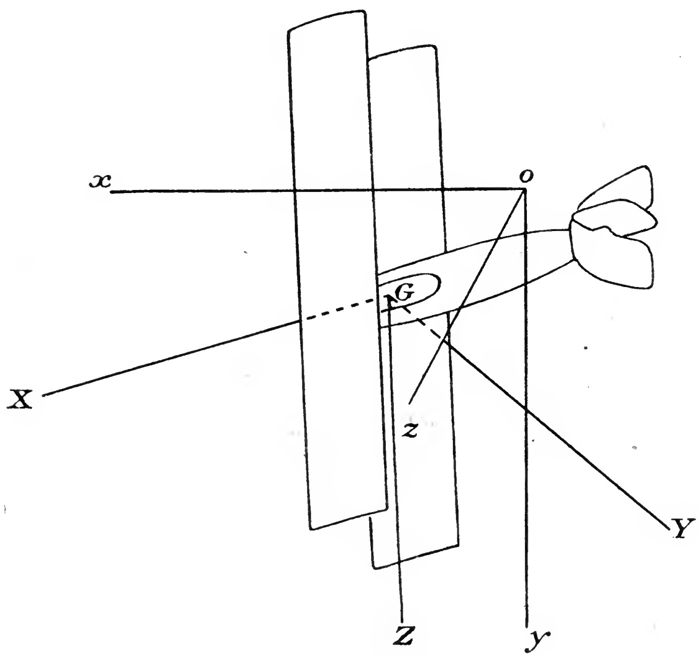

Fig. 49.—Aeroplane Motion in Three Dimensions. Space Axes and Body Axes.

The axes in space to which the motions will be referred are the axes Ox, y, z, defined in § 42, Fig. 9, so that if we have an ordinary right-handed screw with its axis along the axis of x, a rotation from y to z, *i.e.* to the right, will move the screw along the positive direction of the x axis. We have, in fact, a positive set of axes.

As in the two-dimensional work, we must use axes fixed in the body. Not only are the air resistances given by the motions of the body relatively to itself, but also the dynamical specification of the body, namely, the moments and products of inertia would vary with the motion if we used axes fixed in space.

Take then, Fig. 49, the axes GX, GY already defined (the longitudinal and the normal axes) and add a third axis GZ perpendicular to the XY plane, bearing the same relation to this plane as Oz bears to the xy plane. It is thus on the left of the pilot. An aeroplane in which the controls are in neutral positions has a plane of symmetry, which we have used for the vertical plane of motion in the two-dimensional work. We continue to use this plane for the XY plane, although in the three-dimensional motion the symmetry is in general upset, both by turning the **rudder**, which is part of the tail, like the elevator, and by working the **ailerons** or **flaps** in the wings. The changes thus introduced in the machine are small compared to the machine as a whole, and it is an advantage to use the same axes as in two-dimensional work, whilst preserving most of the symmetry in the three-dimensional work.

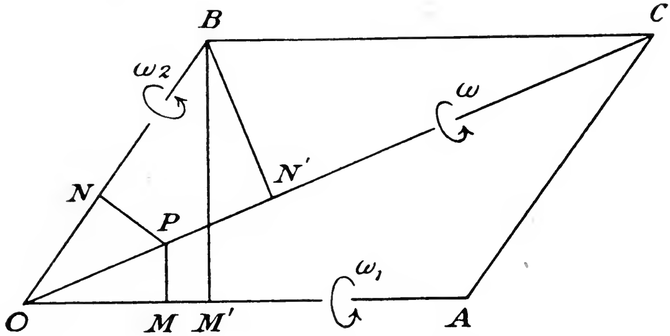

FIG. 50.—Composition of Angular Velocities.

In the longitudinal motion we defined the position of the aeroplane by means of the position of the centre of gravity and the direction of the axis GX. In three dimensions we use, similarly, the three co-ordinates of the centre of gravity referred to the fixed axes Ox, Oy, Oz, and the directions of the three lines GX, GY, GZ, also referred to these fixed axes. If we find the position of the centre of gravity and the directions of the axes GX, GY, GZ from time to time during the motion, we shall have a complete statement of the history of the aeroplane under consideration.

We, therefore, define the motion by calling U_1, U_2, U_3 the velocity components of the centre of gravity along the instantaneous directions of GX, GY, GZ, and Ω_1, Ω_2, Ω_3 the angular velocities of the body about these instantaneous directions. There is no difficulty about the former, as we are well familiar with the vectorial nature of translational velocities, even in three dimensions. But the student may not at once realise that angular velocities can also be treated in the same way; we, therefore, give a short proof, especially as the theorem only holds for angular *velocities* and not for *finite* angular *displacements*.

99. **Angular Velocities are Vectors.**—If a body has angular velocities ω_1, ω_2 about the two intersecting axes OA, OB, Fig. 50, draw OA,

OB proportional to ω_1, ω_2 in such directions that the rotations appear to be clockwise when seen from OA, OB. Complete the parallelogram $OACB$. Take any point P on OC. The motion of P is given by the resultant of two motions, the rotation about OA and the rotation about OB. Let the motion go on for a time δt. If we draw PM, PN perpendicular to OA, OB respectively, then, due to ω_1, the point P moves $PM\omega_1\,\delta t$ normally to and *out of* the plane of the paper, and, due to ω_2, it moves $PN\omega_2\,\delta t$ normally to and *into* the plane of the paper. The total effect is $(PM\omega_1 - PN\omega_2)\,\delta t$ *out of* the plane of the paper. But $PM\omega_1 = PN\omega_2$, since for any point P on the diagonal PC the triangles POA, POB are equal in area, and $OA/OB = \omega_1/\omega_2$. Hence P does not move at all during the interval δt; in other words, the total rotation is such that P is at rest. The same applies to all points on the line OC, so that the motion of the body during δt must be a rotation about OC.

To find the amount of this rotation consider the point B. If BM' is the perpendicular to OA, BN' to OC, then the motion of B is either $\omega_1 BM'\,\delta t$ normally to and *out of* the plane of the paper, or $\omega\, BN'\,\delta t$ in the same direction, where ω is the angular velocity about OC, seen clockwise from OC. Hence $\omega/\omega_1 = BM'/BN' = OC/OA$, since the areas of the triangles BOA, BOC are equal.

Thus the instantaneous effect of the two angular velocities ω_1 about OA and proportional to OA, ω_2 about OB and proportional to OB, is an angular velocity ω about OC proportional to OC, where OC is the diagonal of the parallelogram on OA, OB as adjacent sides. We have then the fundamental vector property of angular velocities. It follows, without any further argument, that any number of angular velocities about lines through a fixed point can be compounded and resolved vectorially.

But the motion of a rigid body can be defined by first fixing, at any instant, the position of a point in it, say the centre of gravity, and then the directions of three lines (given in the body) which pass through this point. Hence the motion can be defined by the velocity components along GX, GY, GZ, and the angular velocity components about GX, GY, GZ.

100. The Physical Argument of the General Method of Moving Axes.—We must use moving axes in the rigid dynamics of three dimensions, just as it was found advisable to do in the two-dimensional work. The difficulties of the subject are correspondingly increased. On the other hand, the simplifications introduced both in the air pressures and in the moments and products of inertia are so great that we can well afford to allow these difficulties.

The argument of the method is as follows. As far as ordinary mechanics is concerned, we must choose axes *fixed in space* for writing down the equations of motion. But we cannot use axes actually fixed in space for all time. We, therefore, say: Let us use for any short interval of time a set of axes which are, for this interval, fixed in space, but which happen, at the beginning of the interval, to coincide with the convenient set of body axes we have defined. Having found the motion with these temporarily fixed axes, we now discard them, and during the next interval of time we use a new set which are, during this interval, fixed in space, but which happen, at the beginning of this interval, to coincide with the new positions of the body axes thus found. In this way we proceed from interval to interval, with the result that after an indefinitely large number of intervals we have executed a finite motion. The process can be

compared to that of climbing up a series of slopes, using the same ladder for each slope, the ladder being dragged up from slope to slope. It is thus seen that the term "moving axes" is really somewhat misleading, as *the axes do not move during the interval when they are used for the equations of motion.*

101. **Equations of Motion of a Rigid Body in Three Dimensions; D'Alembert's Principle.**—Now consider the equations of motion of a rigid body referred to any set of axes fixed in space, OX, Y, Z. The rigid body can be taken to consist of an indefinitely large number of particles rigidly connected, so that there are an indefinitely large number of internal reactions. Let m_i be the mass of a particle which is at any instant at the point (x_i, y_i, z_i), Fig. 51, and let the forces acting on the particle be taken in two parts:

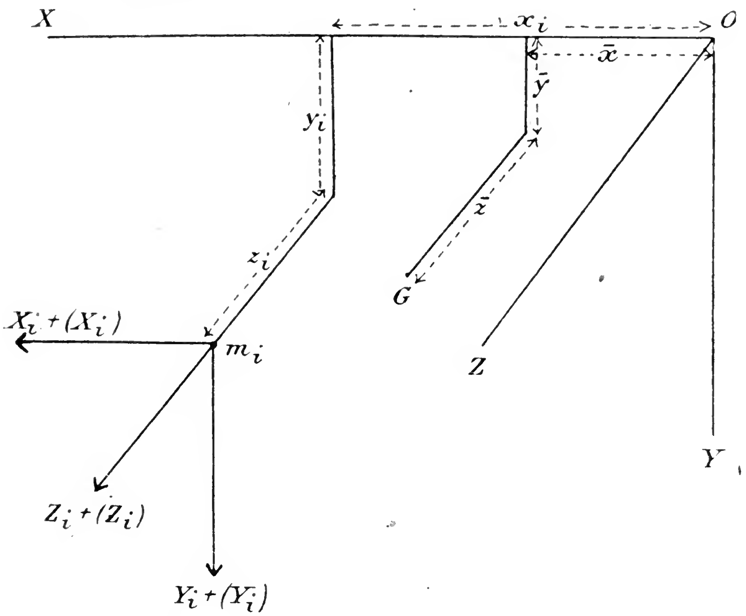

FIG. 51.—Particle under Internal and External Forces: D'Alembert's Principle.

(i) The *internal* forces which we represent by the components (X_i), (Y_i), (Z_i); and

(ii) The *external* forces with components X_i, Y_i, Z_i. The equations of motion of the particle are, therefore,

$$\left.\begin{aligned} m_i\frac{d^2x_i}{dt^2} &= X_i + (X_i), \\ m_i\frac{d^2y_i}{dt^2} &= Y_i + (Y_i), \\ m_i\frac{d^2z_i}{dt^2} &= Z_i + (Z_i). \end{aligned}\right\} \quad \quad (129)$$

If there are n particles, there will be $3n$ such equations.

The internal forces are unknown—they depend on the stresses set up

in the material, and in our present state of advance in mechanics we cannot hope to be able to give very definite statements about them. But in reality we do not need to know them *a priori*. In fact, we can discover them by first solving the motion, after which the internal forces are given by

$$(X_i) = m_i \frac{d^2x_i}{dt^2} - X_i, \quad (Y_i) = m_i \frac{d^2y_i}{dt^2} - Y_i, \quad (Z_i) = m_i \frac{d^2z_i}{dt^2} - Z_i;$$

and to solve the motion we use the fact that *the internal forces*, whatever they may be, *must be a set of forces in equilibrium*. A set of forces in equilibrium must satisfy six conditions—three determined by the fact that they have no *resultant force*, and three by the fact that they have no *resultant moment*. The first three give us

$$\Sigma(X_i) = 0, \quad \Sigma(Y_i) = 0, \quad \Sigma(Z_i) = 0;$$

the second three conditions give, by taking moments about the three axes respectively,

$$\Sigma\{y_i(Z_i) - z_i(Y_i)\} = 0, \quad \Sigma\{z_i(X_i) - x_i(Z_i)\} = 0, \quad \Sigma\{x_i(Y_i) - y_i(X_i)\} = 0.$$

We thus get six, and only six, pieces of information about the motions of the particles of the rigid body. They are

$$\left.\begin{aligned} \Sigma m_i \frac{d^2x_i}{dt^2} = \Sigma X_i, \quad \Sigma m_i \frac{d^2y_i}{dt^2} &= \Sigma Y_i, \quad \Sigma m_i \frac{d^2z_i}{dt^2} = \Sigma Z_i; \\ \Sigma m_i \left(y_i \frac{d^2z_i}{dt^2} - z_i \frac{d^2y_i}{dt^2} \right) &= \Sigma(y_i Z_i - z_i Y_i), \\ \Sigma m_i \left(z_i \frac{d^2x_i}{dt^2} - x_i \frac{d^2z_i}{dt^2} \right) &= \Sigma(z_i X_i - x_i Z_i), \\ \Sigma m_i \left(x_i \frac{d^2y_i}{dt^2} - y_i \frac{d^2x_i}{dt^2} \right) &= \Sigma(x_i Y_i - y_i X_i). \end{aligned}\right\} \quad . \quad . \quad . \quad (130)$$

The important thing about these six equations is that they suffice to determine the motion of the *rigid* body, since this motion is defined by six quantities, say the three velocity components of the centre of gravity, and the three angular velocity components about three known lines through the centre of gravity.

102. **Origin at the Centre of Gravity.**—Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity, and let x_i', y_i', z_i' be the co-ordinates of the particle m_i referred to axes with their origin at the centre of gravity and with their directions GX', GY', GZ' parallel to the original axes Ox, Oy, Oz. Using the facts that

$$x_i = \bar{x} + x_i', \quad y_i = \bar{y} + y_i', \quad z_i = \bar{z} + z_i',$$

and that

$$\Sigma m_i x_i' = \Sigma m_i y_i' = \Sigma m_i z_i' = 0,$$

by the definition of the centre of gravity, so that we also have

$$\Sigma m_i \frac{dx_i'}{dt} = \Sigma m_i \frac{dy_i'}{dt} = \Sigma m_i \frac{dz_i'}{dt} = 0,$$

$$\Sigma m_i \frac{d^2x_i'}{dt^2} = \Sigma m_i \frac{d^2y_i'}{dt^2} = \Sigma m_i \frac{d^2z_i'}{dt^2} = 0,$$

we readily transform the equations (130) into

$$\left.\begin{aligned}
&m\frac{d^2\bar{x}}{dt^2}=\Sigma X_i,\quad m\frac{d^2\bar{y}}{dt^2}=\Sigma Y_i,\quad m\frac{d^2\bar{z}}{dt^2}=\Sigma Z_i\,;\\
&\Sigma m_i\left(y_i'\frac{d^2z_i'}{dt^2}-z_i'\frac{d^2y_i'}{dt^2}\right)=\Sigma(y_i'Z_i-z_i'Y_i),\\
&\Sigma m_i\left(z_i'\frac{d^2x_i'}{dt^2}-x_i'\frac{d^2z_i'}{dt^2}\right)=\Sigma(z_i'X_i-x_i'Z_i),\\
&\Sigma m_i\left(x_i'\frac{d^2y_i'}{dt^2}-y_i'\frac{d^2x_i'}{dt^2}\right)=\Sigma(x_i'Y_i-y_i'X_i);
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (131)$$

where m is the mass of the whole of the rigid body, and the expressions on the right-hand sides are the components of force and components of couple of the whole set of external forces referred to the new axes through

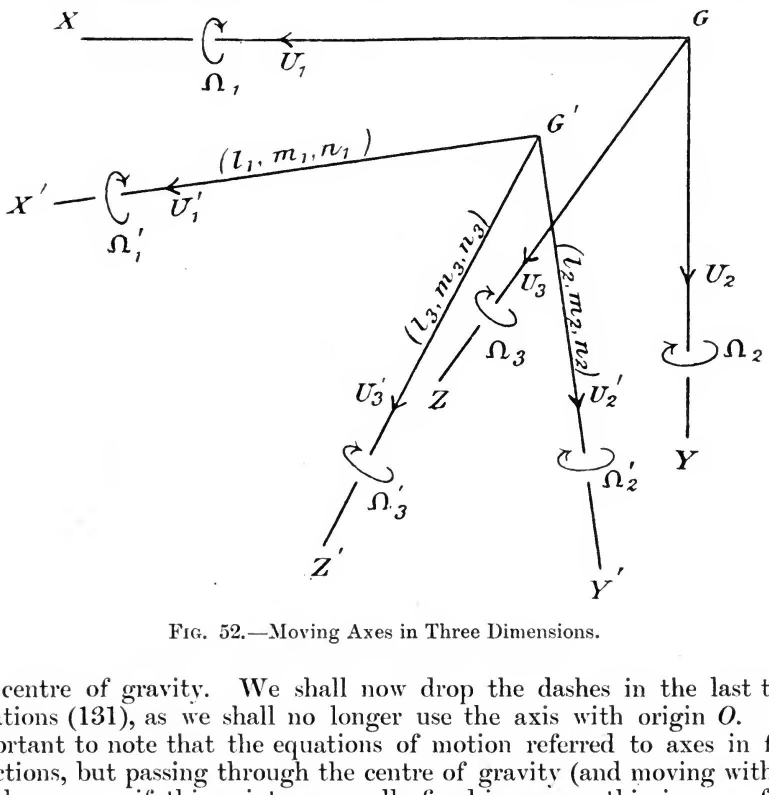

FIG. 52.—Moving Axes in Three Dimensions.

the centre of gravity. We shall now drop the dashes in the last three equations (131), as we shall no longer use the axis with origin O. It is important to note that the equations of motion referred to axes in fixed directions, but passing through the centre of gravity (and moving with it), are the same as if this point were really fixed in space; this is one of the most useful properties of the centre of gravity.

103. **Acceleration̲s Referred to "Moving" Axes.**—We must now discover expressions for

$$\frac{d^2\bar{x}}{dt^2},\quad \frac{d^2\bar{y}}{dt^2},\quad \frac{d^2\bar{z}}{dt^2},\quad \frac{d^2x_i}{dt^2},\quad \frac{d^2y_i}{dt^2},\quad \frac{d^2z_i}{dt^2}$$

in terms of the motions of the axes GX, GY, GZ. It must be remembered that these quantities should be referred to axes fixed in space for a short interval of time.

Let U_1, U_2, U_3 be the components of velocity of the centre of gravity G along the directions GX, GY, GZ; and Ω_1, Ω_2, Ω_3 the angular velocity components of the body about these axes. Let the motion go on for an nterval of time δt, so that G goes to G' and the axes carried by the body go to $G'X'$, $G'Y'$, $G'Z'$ (Fig. 52): the axes to which the motion is referred are to remain at GX, GY, GZ. Suppose that the new components of velocity of the centre of gravity referred to the new axes are U_1', U_2', U_3'; and the new components of angular velocity of the body referred to these axes are Ω_1', Ω_2', Ω_3'. If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the new axes referred to the old ones, then the new velocity components along GX, GY, GZ, *i.e.* the old directions of the axes, are

$$l_1U_1' + l_2U_2' + l_3U_3', \quad m_1U_1' + m_2U_2' + m_3U_3', \quad n_1U_1' + n_2U_2' + n_3U_3'.$$

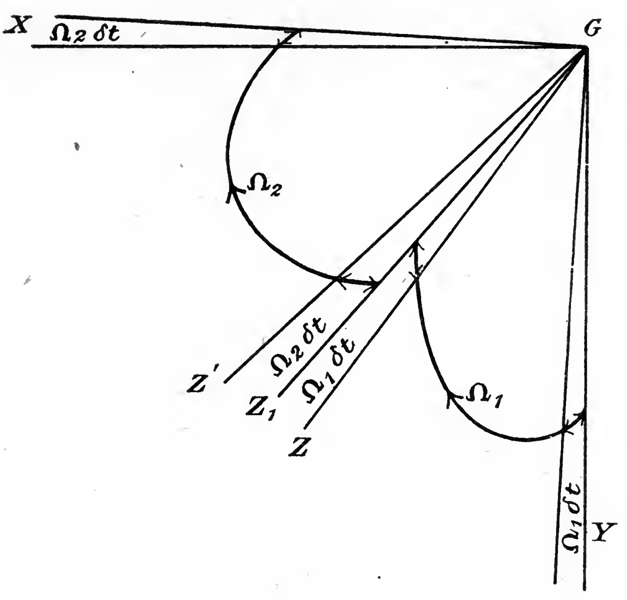

FIG. 53.—Consecutive Positions of Body Axes.

The changes in these components during the interval of time δt are, therefore,

$$(l_1U_1' - U_1) + l_2U_2' + l_3U_3', \quad m_1U_1' + (m_2U_2' - U_2) + m_3U_3',$$
$$n_1U_1' + n_2U_2' + (n_3U_3' - U_3).$$

But the directions of the new axes are obtained from the old one by means of rotations $\Omega_1\delta t$, $\Omega_2\delta t$, $\Omega_3\delta t$ about them. Let us consider the effect on $G'Z'$, say (Fig. 53). The effect of $\Omega_1\delta t$ about GX is to turn GZ through an angle $\Omega_1\delta t$ in the plane YZ; the effect of $\Omega_2\delta t$ about GY is to turn this new line GZ_1, in the new XZ_1 plane, through an angle $\Omega_2\delta t$. Also $\Omega_3\delta t$ has no effect on GZ. Hence, to the first order of the small quantity δt, we can say that

$$n_3 = 1, \quad m_3 = -\Omega_1\delta t, \quad l_3 \doteq \Omega_2\delta t,$$

with similar results for the other axes. Hence

$$\begin{array}{lll} l_1 = 1, & m_1 = \Omega_3 \delta t, & n_1 = -\Omega_2 \delta t; \\ l_2 = -\Omega_3 \delta t, & m_2 = 1, & n_2 = \Omega_1 \delta t; \\ l_3 = \Omega_2 \delta t, & m_3 = -\Omega_1 \delta t, & n_3 = 1; \end{array}$$

so that in time δt the changes in the velocity components referred to the fixed axes GX, GY, GZ are, to the first order,

$$(U_1' - U_1) - \Omega_3 U_2 \delta t + \Omega_2 U_3 \delta t, \quad (U_2' - U_2) - \Omega_1 U_3 \delta t + \Omega_3 U_1 \delta t,$$
$$(U_3' - U_3) - \Omega_2 U_1 \delta t + \Omega_1 U_2 \delta t,$$

since U_1', U_2', U_3' differ from U_1, U_2, U_3 by small quantities. Thus the components of acceleration of the centre of gravity, referred to the axes GX, GY, GZ, which we have taken as fixed in space during the interval δt, are

$$\frac{dU_1}{dt} - \Omega_3 U_2 + \Omega_2 U_3, \quad \frac{dU_2}{dt} - \Omega_1 U_3 + \Omega_3 U_1, \quad \frac{dU_3}{dt} - \Omega_2 U_1 + \Omega_1 U_2. \quad . \ (132)$$

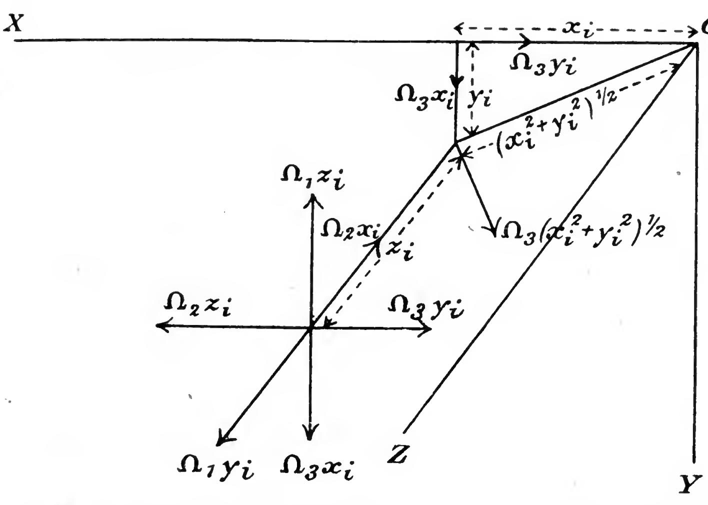

FIG. 54.—Motion of any Particle of a Rigid Body relatively to the Centre of Gravity.

We can use these in the equations of motion for the centre of gravity, if we equate each to the corresponding component of force per unit mass acting on the whole of the body.

104. **Acceleration of any Point in the Body.**—To find correct expressions for d^2x_i/dt^2, d^2y_i/dt^2, d^2z_i/dt^2 for any point whose co-ordinates are (x_i, y_i, z_i) referred to the axes GX, Y, Z in the body, we proceed in a similar way, although the algebra is longer. The calculations are simpler for the centre of gravity, because, as the origin is at G itself, its co-ordinates are always (0, 0, 0). This is not the case for any other point in the body.

Let us first write down the velocity components of (x_i, y_i, z_i) referred to GX, Y, Z. The X component is the relative X velocity of (x_i, y_i, z_i) due to the rotation of the body. The rotation in a short interval δt is $\Omega_1 \delta t$ about GX, $\Omega_2 \delta t$ about GY, $\Omega_3 \delta t$ about GZ. The first gives no displacement parallel to the X axis; the second gives a displacement $z_i \Omega_2 \delta t$ parallel to the X axis; the third gives $-y_i \Omega_3 \delta t$ parallel to the X

axis (Fig. 54). Hence the X velocity of (x_i, y_i, z_i) relatively to the centre of gravity is

$$\left.\begin{aligned}&\frac{dx_i}{dt} = \Omega_2 z_i - \Omega_3 y_i.\\ &\text{Similarly, the other (i.e. the } Y \text{ and } Z\text{) components are}\\ &\frac{dy_i}{dt} = \Omega_3 x_i - \Omega_1 z_i,\ \frac{dz_i}{dt} = \Omega_1 y_i - \Omega_2 x_i.\end{aligned}\right\} \quad \ldots \quad (133)$$

We have then the velocity components of x_i, y_i, z_i referred to GX, GY, GZ.

Now the motion has taken place for a short interval δt, and meanwhile the axes have themselves assumed a new position $G'X', Y', Z'$. There are now new velocity components of (x_i, y_i, z_i). Referred to the new directions, they will be

$$\Omega_2' z_i - \Omega_3' y_i,\quad \Omega_3' x_i - \Omega_1' z_i,\quad \Omega_1' y_i - \Omega_2' x_i,$$

where $\Omega_1', \Omega_2', \Omega_3'$ are the new components of angular velocity of the body. Hence the new velocity components of (x_i, y_i, z_i) referred to the old directions GX, Y, Z are

$$\begin{aligned}&l_1\ (\Omega_2' z_i - \Omega_3' y_i) + l_2\ (\Omega_3' x_i - \Omega_1' z_i) + l_3\ (\Omega_1' y_i - \Omega_2' x_i),\\ &m_1 (\Omega_2' z_i - \Omega_3' y_i) + m_2 (\Omega_3' x_i - \Omega_1' z_i) + m_3 (\Omega_1' y_i - \Omega_2' x_i),\\ &n_1\ (\Omega_2' z_i - \Omega_3' y_i) + n_2\ (\Omega_3' x_i - \Omega_1' z_i) + n_3\ (\Omega_1' y_i - \Omega_2' x_i),\end{aligned}$$

where $l_1, m_1, n_1, \ldots$ have the same meanings as in the last article. Referred to the axes GX, GY, GZ, which are for the time δt considered to be fixed in space, we have for the X acceleration of (x_i, y_i, z_i) the equation

$$\frac{d^2x_i}{dt^2}\delta t = \{l_1(\Omega_2' z_i - \Omega_3' y_i) - (\Omega_2 z_i - \Omega_3 y_i)\} + l_2(\Omega_3' x_i - \Omega_1' z_i) + l_3(\Omega_1' y_i - \Omega_2' x_i),$$

with similar equations for the other components. For a very short interval δt we use the values of $l_1, m_1, n_1, \ldots$ given in § 103. We, therefore, get

$$\left.\begin{aligned}&\frac{d^2x_i}{dt^2} = z_i\frac{d\Omega_2}{dt} - y_i\frac{d\Omega_3}{dt} - (\Omega_2^2 + \Omega_3^2)x_i + \Omega_1\Omega_2 y_i + \Omega_1\Omega_3 z_i.\\ &\text{Similarly,}\\ &\frac{d^2y_i}{dt^2} = x_i\frac{d\Omega_3}{dt} - z_i\frac{d\Omega_1}{dt} - (\Omega_3^2 + \Omega_1^2)y_i + \Omega_2\Omega_3 z_i + \Omega_2\Omega_1 x_i,\\ &\text{and}\\ &\frac{d^2z_i}{dt^2} = y_i\frac{d\Omega_1}{dt} - x_i\frac{d\Omega_2}{dt} - (\Omega_1^2 + \Omega_2^2)z_i + \Omega_3\Omega_1 x_i + \Omega_3\Omega_2 y_i.\end{aligned}\right\} \quad \ldots \quad (134)$$

105. **Rates of Change of the Angular Momenta about the Centre of Gravity.**—Substitute the values (134) in the equations (131) (with the dashes removed). We get that the sum of the moments of all the external forces about the axis GX must be equal to

$$\begin{aligned}\Sigma m_i\{(y_i^2 + z_i^2)\frac{d\Omega_1}{dt} - x_i y_i\frac{d\Omega_2}{dt} - x_i z_i\frac{d\Omega_3}{dt}\\ + y_i z_i(\Omega_3^2 - \Omega_2^2) - x_i z_i\,\Omega_1\Omega_2 + (y_i^2 - z_i^2)\,\Omega_2\Omega_3 + x_i y_i\,\Omega_1\Omega_3\},\end{aligned}$$

with similar expressions for the moments about the axes GY and GZ. If we introduce the moments and products of inertia given by

$$\begin{aligned}&A = \Sigma m_i(y_i^2 + z_i^2),\quad B = \Sigma m_i(z_i^2 + x_i^2),\quad C = \Sigma m_i(x_i^2 + y_i^2),\\ &D = \Sigma m_i y_i z_i,\qquad E = \Sigma m_i z_i x_i,\qquad F = \Sigma m_i x_i y_i.\end{aligned}$$

then the sums of the moments of all the external forces about GX, GY, GZ must be equal respectively to

$$\left.\begin{aligned}
&A\frac{d\Omega_1}{dt}-F\frac{d\Omega_2}{dt}-E\frac{d\Omega_3}{dt}-D(\Omega_2{}^2-\Omega_3{}^2)-E\Omega_1\Omega_2+F\Omega_3\Omega_1+(C-B)\Omega_2\Omega_3,\\
&B\frac{d\Omega_2}{dt}-D\frac{d\Omega_3}{dt}-F\frac{d\Omega_1}{dt}-E(\Omega_3{}^2-\Omega_1{}^2)-F\Omega_2\Omega_3+D\Omega_1\Omega_2+(A-C)\Omega_3\Omega_1,\\
&C\frac{d\Omega_3}{dt}-E\frac{d\Omega_1}{dt}-D\frac{d\Omega_2}{dt}-F(\Omega_1{}^2-\Omega_2{}^2)-D\Omega_3\Omega_1+E\Omega_2\Omega_3+(B-A)\Omega_1\Omega_2.
\end{aligned}\right\} \quad (135)$$

These expressions can be simplified by the introduction of the notation

$$H_1=A\Omega_1-F\Omega_2-E\Omega_3,\quad H_2=B\Omega_2-D\Omega_3-F\Omega_1,\quad H_3=C\Omega_3-E\Omega_1-D\Omega_2,$$

where H_1, H_2, H_3 are the angular momenta of the body about the axes of X, Y, Z, defined as

$$H_1=\Sigma m_i\left(y_i\frac{dz_i}{dt}-z_i\frac{dy_i}{dt}\right),\quad H_2=\Sigma m_i\left(z_i\frac{dx_i}{dt}-x_i\frac{dz_i}{dt}\right),\quad H_3=\Sigma m_i\left(x_i\frac{dy_i}{dt}-y_i\frac{dx_i}{dt}\right).$$

The expressions (135) now become

$$\frac{dH_1}{dt}-\Omega_3H_2+\Omega_2H_3,\quad \frac{dH_2}{dt}-\Omega_1H_3+\Omega_3H_1,\quad \frac{dH_3}{dt}-\Omega_2H_1+\Omega_1H_2.$$

These are, in fact, the time differential coefficients of the angular momenta referred to the axes GX, GY, GZ considered fixed in space for the short interval of time δt—a special case of a general theorem on the variation of vectors referred to moving axes.

106. **Components of Air Force and Couple; The Gravity Components.**—It only remains to write down the components of force and couple due to gravity, the propeller thrust, and the air resistance. We assume the propeller thrust to pass through the centre of gravity, and the X axis to be along the propeller axis. The components of force due to the thrust and air resistance we take to be T along the positive direction of the X axis, and R_1, R_2, R_3 along the negative directions of the X, Y, Z axes, each measured per unit mass of the machine. The components of couple due to the air resistance we take to be AG_1, BG_2, CG_3 for the whole machine along the negative senses of Ω_1, Ω_2, Ω_3; thus G_1 is the moment about the X axis from Z to Y of the air resistance per unit moment of inertia about the axis of X; etc.

The components of gravity are not so easy to write down, since gravity acts in a direction which is fixed once for all, whereas GX, GY, GZ assume different directions as the motion goes on. It is necessary to devise some means of writing down the actual directions of GX, GY, GZ at any moment, referred to Ox, Oy, Oz.

We have already said that GX, GY, GZ have such relative directions that when G is at O and GX along Ox, GY along Oy, then GZ is along Oz. Let us then define the position of the body by stating how it has to be rotated so as to put the axes that are originally coincident (in direction) with Ox, Oy, Oz into the actual directions GX, GY, GZ. As we are only concerned with directions we can imagine G to be at O.

In Fig. 55 the axes X, Y, Z are coincident with x, y, z. Let there now be a **yaw** through an angle Ψ about the axis of Y. Of the new axes OX_1, OY_1, OZ_1, the axis OY_1 is at OY, but OX goes to OX_1, OZ to OZ_1, where the angles XOX_1, ZOZ_1 are each Ψ, X_1Z_1 remaining in the plane of XZ.

Now let the machine **pitch** through an angle Θ about the axis of Z_1. In Fig. 56 of the new axes OX_2, OY_2, OZ_2, the axis OZ_2 is at OZ_1, but OX_1, OY_1 each turns through an angle Θ to OX_2, OY_2, both remaining in the plane of X_1Y_1.

Finally, let there be a **roll** through an angle Φ about the axis of X_2. In Fig. 57, of the new axes OX_3, OY_3, OZ_3, the axis OX_3 is at OX_2, but

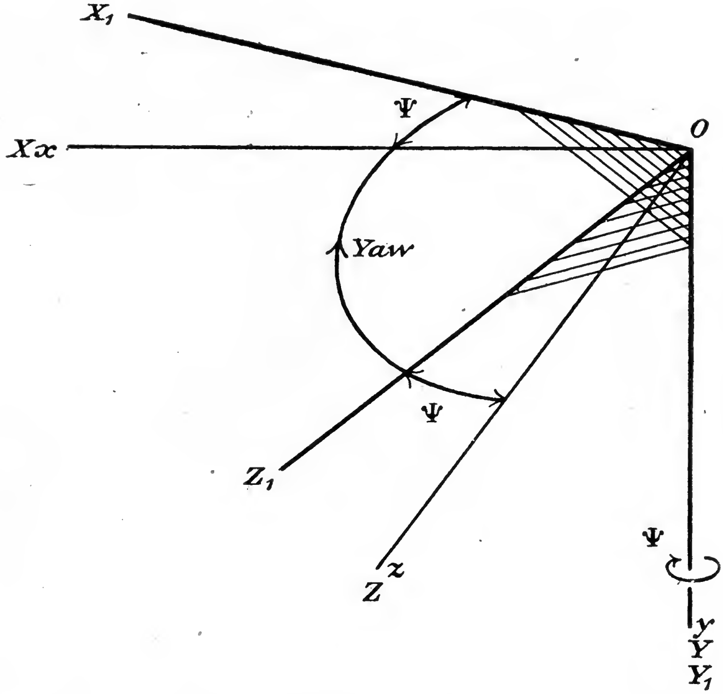

Fig. 55.—Yaw (Ψ) about the axis of Y.

OY_2, OZ_2 each turns through an angle Φ to OY_3, OZ_3, the latter remaining in the plane of Y_2Z_2.

The axes OX_3, OY_3, OZ_3 are the directions of the body axes GX, GY, GZ at any moment. It is important to remember that the vertical plane through the X axis makes an angle Ψ with the vertical xy plane. It must also be borne in mind that the order of the rotations is by no means commutative, as when we dealt with angular velocities. Angular displacements are only commutative when they are infinitely small. But when they are finite we must adopt some standard way of carrying them out. In the present work we are using the natural order of first changing the azimuth (defined by Ψ) and then manipulating the mode of flight of the machine by means of pitching and rolling.

The components of gravity along OX_1, OY_1, OZ_1 in Fig. 55 are

$$0, \; g, \; 0;$$

the components along OX_2, OY_2, OZ_2 in Fig. 56 are, therefore,

$$g \sin \Theta, \; g \cos \Theta, \; 0;$$

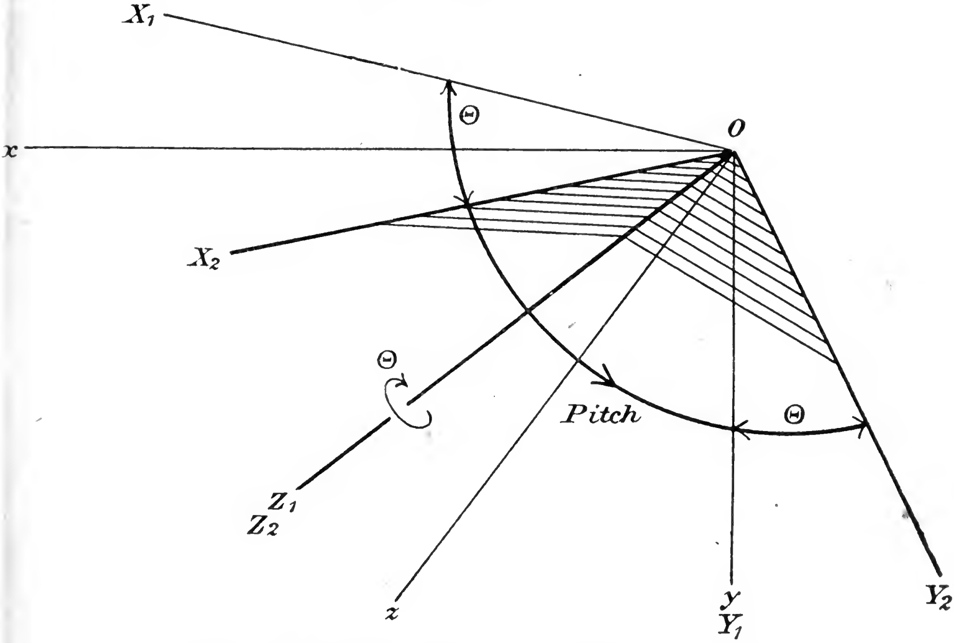

FIG. 56.—Pitch (Θ) about the axis of Z, after the Yaw in Fig. 55.

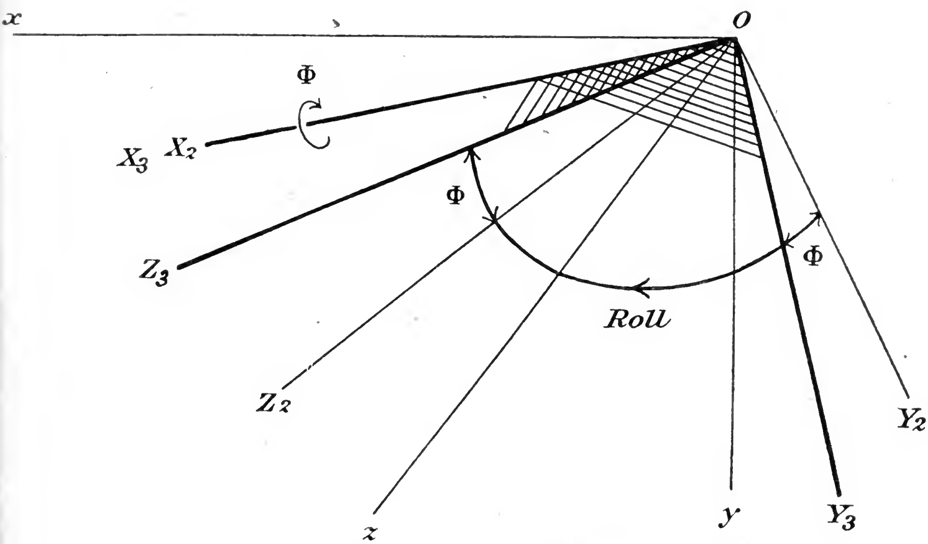

FIG. 57.—Roll (Φ) about the axis of X, after the Pitch in Fig. 56.

so that the components along OX_3, OY_3, OZ_3 in Fig. 57 are

$$g \sin \Theta, \quad g \cos \Theta \cos \Phi, \quad -g \cos \Theta \sin \Phi.$$

These are the components along the directions of the body axes GX, GY, GZ.

107. **Equations of Motion.**—The equations of motion are, therefore,

$$\left.\begin{aligned}
&\frac{dU_1}{dt} - \Omega_3 U_2 + \Omega_2 U_3 = g \sin \Theta + T - R_1, \\
&\frac{dU_2}{dt} - \Omega_1 U_3 + \Omega_3 U_1 = g \cos \Theta \cos \Phi - R_2, \\
&\frac{dU_3}{dt} - \Omega_2 U_1 + \Omega_1 U_2 = - g \cos \Theta \sin \Phi - R_3. \\
&A\frac{d\Omega_1}{dt} - F\frac{d\Omega_2}{dt} - E\frac{d\Omega_3}{dt} - D(\Omega_2{}^2 - \Omega_3{}^2) - E\Omega_1\Omega_2 + F\Omega_3\Omega_1 + (C-B)\Omega_2\Omega_3 = -AG_1, \\
&B\frac{d\Omega_2}{dt} - D\frac{d\Omega_3}{dt} - F\frac{d\Omega_1}{dt} - E(\Omega_3{}^2 - \Omega_1{}^2) - F\Omega_2\Omega_3 + D\Omega_1\Omega_2 + (A-C)\Omega_3\Omega_1 = -BG_2, \\
&C\frac{d\Omega_3}{dt} - E\frac{d\Omega_1}{dt} - D\frac{d\Omega_2}{dt} - F(\Omega_1{}^2 - \Omega_2{}^2) - D\Omega_3\Omega_1 + E\Omega_2\Omega_3 + (B-A)\Omega_1\Omega_2 = -CG_3.
\end{aligned}\right\} \quad (13$$

108. **Angular Velocities in Space.**—But we are not yet in a position to attempt solutions, because Ω_1, Ω_2, Ω_3 are referred to the body axes GX, GY, GZ, whilst Θ, Φ, Ψ are referred to the axes Ox, Oy, Oz. We must now establish the values of Ω_1, Ω_2, Ω_3 in terms of the changes in Θ, Φ, Ψ.

Suppose that we have additional rotations

$$\frac{d\Theta}{dt}\delta t, \quad \frac{d\Phi}{dt}\delta t, \quad \frac{d\Psi}{dt}\delta t.$$

The rotation $\frac{d\Psi}{dt}\delta t$ gives (Fig. 55) this rotation about OY_1 and nothing about OX_1, OZ_1; therefore (Fig. 56), it gives rotations

$$\frac{d\Psi}{dt}\delta t \,.\, \sin \Theta \text{ about } OX_2, \quad \frac{d\Psi}{dt}\delta t \,.\, \cos \Theta \text{ about } OY_2, \quad \text{nothing about } OZ_2;$$

and finally (Fig. 57), we get

$$\frac{d\Psi}{dt}\delta t \,.\, \sin \Theta \text{ about } OX_3, \quad \frac{d\Psi}{dt}\delta t \,.\, \cos \Theta \cos \Phi \text{ about } OY_3,$$
$$-\frac{d\Psi}{dt}\delta t \,.\, \cos \Theta \sin \Phi \text{ about } OZ_3.$$

Thus an angular velocity $d\Psi/dt$ contributes to Ω_1, Ω_2, Ω_3 amounts

$$\frac{d\Psi}{dt}\sin \Theta, \quad \frac{d\Psi}{dt}\cos \Theta \cos \Phi, \quad -\frac{d\Psi}{dt}\cos \Theta \sin \Phi.$$

By similar arguments we find that angular velocities $d\Theta/dt$, $d\Phi/dt$ respectively contribute to Ω_1, Ω_2, Ω_3 the amounts

$$0, \quad \frac{d\Theta}{dt}\sin \Phi. \quad \frac{d\Theta}{dt}\cos \Phi \quad \text{(Figs. 56, 57)};$$

$$\frac{d\Phi}{dt}, \quad 0, \quad 0 \quad \text{(Fig. 57)}.$$

Hence we deduce the relations

$$\left.\begin{aligned}\Omega_1 &= \frac{d\Phi}{dt} + \sin\Theta\frac{d\Psi}{dt},\\ \Omega_2 &= \sin\Phi\frac{d\Theta}{dt} + \cos\Theta\cos\Phi\frac{d\Psi}{dt},\\ \Omega_3 &= \cos\Phi\frac{d\Theta}{dt} - \cos\Theta\sin\Phi\frac{d\Psi}{dt}.\end{aligned}\right\} \quad (137)$$

If we substitute these values of Ω_1, Ω_2, Ω_3 in equations (136), we get six equations for the six quantities U_1, U_2, U_3, Θ, Φ, Ψ, from which the motion can be found, if methods of integration can be devised.

It need hardly be said that we shall not attempt any general solution of these equations, which we shall not even write down explicitly. As in the two-dimensional work, we must be content with some special cases and the theory of stability.

109. **Rectilinear Steady Motion.**—The first special case is that of steady motion in a straight line with no rotation. To find when such is possible, we must, in (136), put U_1, U_2, U_3 constant ($= U_{10}$, U_{20}, U_{30}), and Ω_1, Ω_2, Ω_3 zero, so that Θ, Φ, Ψ are constant (Θ_0, Φ_0, Ψ_0). We get the six conditions

$$\left.\begin{aligned} g\sin\Theta_0 + T_0 &= R_{10},\\ g\cos\Theta_0\cos\Phi_0 &= R_{20},\\ -g\cos\Theta_0\sin\Phi_0 &= R_{30},\\ G_{10} = G_{20} = G_{30} &= 0.\end{aligned}\right\} \quad (138)$$

For a given form of machine the last three conditions *prescribe the direction of relative motion*, which must be such that no moments exist due to air pressures. With given shape of body the ratios of R_{10}, R_{20}, R_{30} are given. Hence Φ_0 and $g\cos\Theta_0$ are found. If now T_0 is given we deduce the speed since the R's depend on the speed. We thus have definite values of Θ_0, Φ_0, and, therefore, definite orientation of the body, and also prescribed velocity in a definite direction in space. It is important to notice that the azimuth, Ψ_0, is not defined in a rectilinear steady motion.

Thus, as in two-dimensional motion, given shape and thrust determine the steady motion completely. The shape depends on the elevator already mentioned in the two-dimensional work, and the rudder and ailerons. If, then, an aeroplane is in a state of steady motion and some change is made in the elevator, rudder, or ailerons, a new steady motion will be appropriate to the new conditions. Further control of the new steady motion lies in the modification of the engine.

We see that for rectilinear steady motion it is not necessary to restrict ourselves to the plane of symmetry.

We can imagine the propeller thrust zero; the steady motion in this case is a glide. Of course, the direction and velocity, as well as the position, of the aeroplane will depend on the positions of the elevator, rudder, and ailerons.

110. **Stability.**—As in the two-dimensional problem, the question now arises: If the actual motion of an aeroplane is slightly different from the appropriate rectilinear steady motion, will the aeroplane tend to assume this steady motion? This is the problem of stability in three

dimensions. In a practical form the question is: If an aeroplane moving steadily in a straight line has sudden changes made in its elevator, rudder, and ailerons, as well as in the condition of the engine, will it settle down ultimately into the new rectilinear steady motion, or will the motion deviate more and more from this state? But, of course, the analysis and results will apply, too, to the case of any small divergence from the rectilinear steady motion appropriate to the contemporary state of an aeroplane, caused by a wind-gust or any other disturbance.

111. **General Stability of the Parachute.**—It will be an advantage (as in the two-dimensional work, Chapter III, §§ 77–84), to

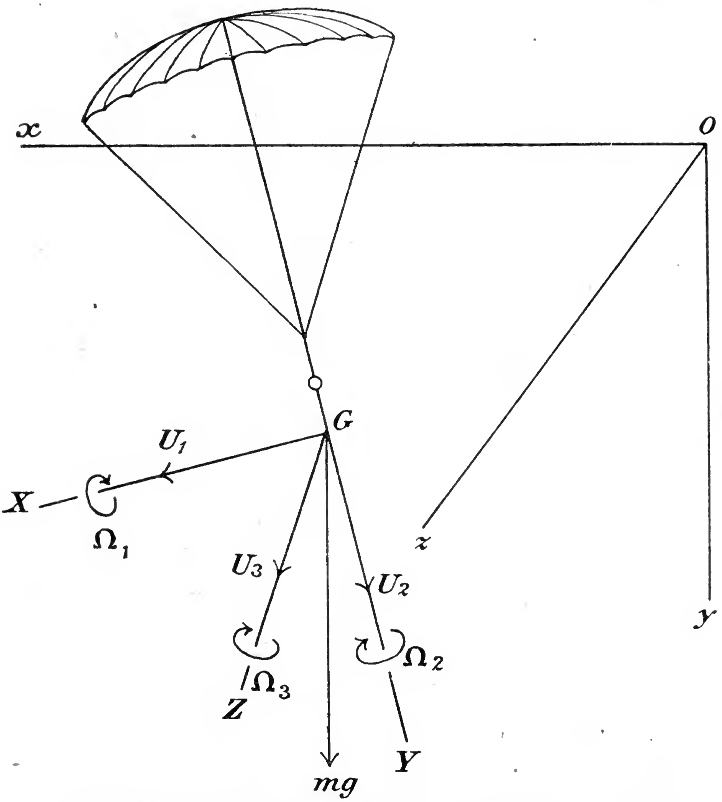

Fig. 58.—Stability of the Parachute in Three Dimensions.

make a short digression here in order to illustrate the type of analysis in the general problem by considering the parachute problem, which is similar to that of the aeroplane, but much simpler. The parachute problem is simpler because of the symmetry. We have, in § 79, taken a vertical plane of motion, without defining exactly how such a vertical plane of motion arises. It would, in fact, be an accident, depending on the disturbance contemplated there *happening* to be such that the motion was uniplanar. In general, no such plane is definable, and we must choose our axes again.

This gives no trouble, because of the symmetry of the parachute round an axis. Take, then (Fig. 58), the downward vertical through the

centre of gravity in the steady fall for the axis GY, and choose any two mutually perpendicular lines GX, GZ in the horizontal plane through the centre of gravity, so that GX, Y, Z form a system of rectangular axes similar to those used in § 98, Fig. 49. We let U_1, U_2, U_3, Ω_1, Ω_2, Ω_3, Θ, Φ, Ψ have the same meanings as in the case of the aeroplane; also R_1, R_2, R_3, G_1, G_2, G_3. There is no thrust T.

Because of the symmetry we now have, for the moments and products of inertia,

$$A = C, \quad D = E = F = 0.$$

The passenger will place himself along the Y axis, so that he causes little divergence from symmetry. The equations of motion (136) become

$$\begin{aligned}
\frac{dU_1}{dt} - \Omega_3 U_2 + \Omega_2 U_3 &= \quad g \sin \Theta - R_1, \\
\frac{dU_2}{dt} - \Omega_1 U_3 + \Omega_3 U_1 &= \quad g \cos \Theta \cos \Phi - R_2, \\
\frac{dU_3}{dt} - \Omega_2 U_1 + \Omega_1 U_2 &= - g \cos \Theta \sin \Phi - R_3, \\
\frac{d\Omega_1}{dt} + \left(1 - \frac{B}{A}\right) \Omega_2 \Omega_3 &= - G_1, \\
\frac{d\Omega_2}{dt} \qquad\qquad\qquad &= - G_2, \\
\frac{d\Omega_3}{dt} - \left(1 - \frac{B}{A}\right) \Omega_1 \Omega_2 &= - G_3.
\end{aligned}$$

The simplification in the momentum equations is very appreciable.

112. In the steady fall we have

$$U_{20} = k, \quad U_{10} = U_{30} = \Omega_{10} = \Omega_{20} = \Omega_{30} = \Theta_0 = \Phi_0 = \Psi_0 = 0,$$

also

$$R_{20} = g, \quad R_{10} = R_{30} = G_{10} = G_{20} = G_{30} = 0.$$

Let us, therefore, use

$$U_1 = u_1, \; U_2 = k + u_2, \; U_3 = u_3, \; \Omega_1 = \omega_1, \; \Omega_2 = \omega_2, \; \Omega_3 = \omega_3, \; \Theta = \theta, \; \Phi = \phi, \; \Psi = \psi,$$

where all the small letters represent small quantities, and let us see whether, by assuming them so small as to have squares and products, etc., negligible, we get consistent equations. If this is the case, we get stability, and no matter how the disturbance from the steady fall is caused, so long as it is small enough, the parachute will tend to resume the steady fall. If this is not the case, the parachute will not tend to steady motion if disturbed, and as disturbances are always taking place in practice, it will not be safe to use the parachute.

The equations of motion, assuming stability, are

$$\left.\begin{aligned}
\frac{du_1}{dt} - k\omega_3 &= \quad g\theta - R_1, \\
\frac{du_2}{dt} \qquad &= \quad g - R_2, \\
\frac{du_3}{dt} + k\omega_1 &= - g\phi - R_3,
\end{aligned}\right\} ; \quad
\left.\begin{aligned}
\frac{d\omega_1}{dt} &= - G_1, \\
\frac{d\omega_2}{dt} &= - G_2, \\
\frac{d\omega_3}{dt} &= - G_3.
\end{aligned}\right\}$$

Now

$$R_1 = R_{10} + k(a_x u_1 + b_x u_2 + c_x u_3 + d_x \omega_1 + e_x \omega_2 + f_x \omega_3)$$

to the first order of small quantities; and we have five similar equations for R_2, R_3, G_1, G_2, G_3. We thus have to consider thirty-six derivatives, which are absolute constants as far as our present problem is concerned. The real simplicity of the parachute problem arises from the fact that many of these derivatives are essentially zero, because of the symmetry. Using arguments similar to those used in § 80, we easily show that

$$b_x = c_x = d_x = e_x = a_y = c_y = d_y = e_y = f_y = a_z = b_z = e_z = f_z$$
$$= a_1 = b_1 = e_1 = f_1 = a_2 = b_2 = c_2 = d_2 = f_2 = b_3 = c_3 = d_3 = e_3 = 0\,;$$

in other words, all the derivatives of R_1, R_2, G_3 with respect to u_3, ω_1, ω_2, and all the derivatives of R_3, G_1, G_2 with respect to u_1, u_2, ω_3, vanish, due to the symmetry about the plane XY, in addition to b_x, a_y, f_y, e_z, e_1, c_2, d_2, b_3, due to the fact that there is also symmetry about the plane YZ. In an aeroplane which has a plane of symmetry—*i.e.* rudder and ailerons in neutral positions—the additional derivatives enumerated do not necessarily vanish. When the symmetry is altogether destroyed, none of the derivatives will vanish *a priori*.

Consider further the equations (137) for the angular velocities of the X, Y, Z axes in terms of Θ, Φ, Ψ. Putting $\Theta = \theta$, $\Phi = \phi$, $\Psi = \psi$, we get, to the first order of small quantities, the relations

$$\omega_1 = \frac{d\phi}{dt}, \quad \omega_2 = \frac{d\psi}{dt}, \quad \omega_3 = \frac{d\theta}{dt}.$$

We can now write the equations of motion for the disturbed parachute (assuming that the small quantities remain small) in the following form:

$$\left.\begin{aligned}
\frac{du_1}{dt} - k\frac{d\theta}{dt} &= g\theta - ka_x u_1 - kf_x\frac{d\theta}{dt},\\
\frac{du_2}{dt} &= -kb_y u_2,\\
\frac{du_3}{dt} + k\frac{d\phi}{dt} &= -g\phi - kc_z u_3 - kd_z\frac{d\phi}{dt},\\
\frac{d^2\phi}{dt^2} &= -kc_1 u_3 - kd_1\frac{d\phi}{dt},\\
\frac{d^2\psi}{dt^2} &= -ke_2\frac{d\psi}{dt},\\
\frac{d^2\theta}{dt^2} &= -ka_3 u_1 - kf_3\frac{d\theta}{dt},
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (139)$$

where a_x, f_x, b_y, c_z, d_z, c_1, d_1, e_2, a_3, f_3 are certain numbers which depend on the parachute in its steady motion.

113. **Splitting up of the Parachute Problem.**—If we examine the equations (139) we notice a remarkable fact. There are six unknown quantities, which are to be determined by these equations and which define the motion of the parachute. These are u_1, u_2, u_3, θ, ϕ, ψ. Now u_2 only occurs in the second equation, and this equation only involves u_2, since b_y is a constant which is presumably known. Also ψ only occurs in the fifth equation, and this equation only involves ψ, since e_2 is a known constant. Thus the second equation determines u_2 completely, and the fifth equation determines ψ completely.

Further, u_1 and θ only occur in the first and sixth equations, and these equations only involve u_1 and θ. Hence u_1 and θ are completely determined by these two equations. Finally, u_3 and ϕ are completely determined by the second and fourth equations.

The disturbed motion is thus given by

$$u_2 = u_{20}e^{-kb_yt}, \quad \frac{d\psi}{dt} = \omega_2 = \omega_{20}e^{-ke_2t}; \quad \ldots \ldots \quad (140)$$

$$\left.\begin{aligned}(D + ka_x)u_1 + (\overline{f_x - 1}.kD - g)\theta &= 0,\\ (\quad ka_3)u_1 + (D^2 + kf_3D \quad)\theta &= 0\end{aligned}\right\}; \quad \ldots \ldots \quad (141)$$

$$\left.\begin{aligned}(D + kc_z)u_3 + (\overline{d_z + 1}.kD + g)\phi &= 0,\\ (\quad kc_1)u_3 + (D^2 + kd_1D \quad)\phi &= 0\end{aligned}\right\}, \quad \ldots \ldots \quad (142)$$

where u_{20} is the value of u_2 at $t = 0$, ω_{20} the value of ω_2 or $d\psi/dt$ at $t = 0$, and $D \equiv d/dt$.

We may assume *a priori* that b_y and e_2 are both positive, so that an increased downward velocity gives an increased upward air pressure, and a rotation about a vertical axis is opposed by a sort of frictional resistance. Thus if u_{20}, ω_{20} are small, u_2, ω_2 are small all through the motion. Instability cannot arise through small changes in downward fall or small vertical rotations.

If, now, we turn our attention to the equations (141) for u_1, θ, we find that they are exactly the same as we obtained in § 81, equation (85), for the two-dimensional disturbance: by definition, the derivatives used have the same values in both the problem as here discussed and the two-dimensional problem. The conditions of stability as regards u_1 and θ are, therefore, as already discussed above, §§ 80–83. But now we expect that the stability conditions for u_3, ϕ should also be the same, since by the symmetry of the parachute we could have put the X axis where we have put the Z axis. This is actually the case. The rotation in the YZ plane is opposite in sense to that in the XY plane in (141), so that by symmetry we have $c_z = a_x$, $c_1 = - a_3$, $d_z = -f_x$, $d_1 = f_3$. If, now, we change u_1 into u_3, θ into $- \phi$, and replace a_x, f_x, a_3, f_3, by c_z, $- d_z$, $- c_1$, d_1, we obtain the equations (142).

The general three-dimensional problem of parachute stability is, therefore, no more complicated than the two-dimensional problem. The student must clearly grasp the physical reason for this. Since the deviation from the steady fall is assumed small, and there is symmetry about the XY plane, anything that takes place in the XY plane does not, to the first order of small quantities, affect the motion in the YZ plane; and *vice versa*, since there is also symmetry about the YZ plane.

114. **The Symmetrical Aeroplane.**—In the case of the aeroplane the general stability problem does not admit of this simplification. But before we approach the general problem let us consider the three-dimensional disturbance of a symmetrical aeroplane, whose steady motion is, therefore, in a vertical plane coincident with the plane of symmetry. We are now to complete the discussion of stability which was only partially considered in Chapter III, § 87. There we assumed the symmetrical aeroplane to have such a disturbance that its disturbed motion

is also longitudinal. Now we are generalising and taking the small disturbance anyhow.

We expect that the symmetry about the plane XY should cause simplification of the process and of the results; any small motion outside this plane should not affect, to the first order of small quantities, the motion in this plane, and *vice versa.*

For the symmetrical aeroplane we have the products of inertia D, E, both zero. In the steady motion we have

$$U_1 = U_{10},\ U_2 = U_{20},\ U_{30} = 0,\ \Theta = \Theta_0,\ \Phi_0 = 0,\ \Psi_0 = 0,\ \Omega_{10} = 0,\ \Omega_{20} = 0,\ \Omega_{30} = 0.$$

Let us then put

$$U_1 = U_{10} + u_1,\ U_2 = U_{20} + u_2,\ U_3 = u_3,\ \Theta = \Theta_0 + \theta,\ \Phi = \phi,\ \Psi = \psi,$$
$$\Omega_1 = \omega_1,\ \Omega_2 = \omega_2,\ \Omega_3 = \omega_3,$$

the small letters denoting small quantities whose squares, products, etc., can be neglected. The equations of motion (136) become

$$\left.\begin{aligned}
&\frac{du_1}{dt} - U_{20}\omega_3 = g(\sin\Theta_0 + \theta\cos\Theta_0) + T - R_1,\\
&\frac{du_2}{dt} + U_{10}\omega_3 = g(\cos\Theta_0 - \theta\sin\Theta_0) - R_2,\\
&\frac{du_3}{dt} - U_{10}\omega_2 + U_{20}\omega_1 = -g\cos\Theta_0 \,.\, \phi - R_3,\\
&\qquad\frac{d\omega_1}{dt} - \frac{F}{A}\frac{d\omega_2}{dt} = -G_1,\\
&\qquad\frac{d\omega_2}{dt} - \frac{F}{B}\frac{d\omega_1}{dt} = -G_2,\\
&\qquad\frac{d\omega_3}{dt} \qquad\qquad = -G_3.
\end{aligned}\right\} \qquad (143)$$

where, by (137),

$$\left.\begin{aligned}
\omega_1 &= \frac{d\phi}{dt} + \sin\Theta_0 \,.\, \frac{d\psi}{dt},\\
\omega_2 &= \qquad\quad \cos\Theta_0 \,.\, \frac{d\psi}{dt},\\
\omega_3 &= \frac{d\theta}{dt}.
\end{aligned}\right\} \qquad (144)$$

115. **Longitudinal and Lateral Stability Separated.** — As already pointed out in the work on the parachute, § 112, the derivatives of R_1, R_2, G_3 with respect to u_3, ω_1, ω_2, and of R_3, G_1, G_2 with respect to u_1, u_2, ω_3, are all zero, because of the symmetry about the XY plane. As regards the propeller thrust, since it has not the symmetry about the plane of XY, we cannot really say that certain of its derivatives vanish. It is, however, found that the only really important derivative of the thrust is that due to u_1, and we shall include it in the X derivative as in § 87. We neglect the gyroscopic effect of the air force on the propeller and that due to the rotation of the engine if it is of the rotary type. We find that the quantities u_1, u_3, θ (which involves ω_3) only occur in the first, second, and sixth equations, and that these equations only include these quantities; also that the quantities u_3, ϕ, ψ (which also involve

ω_1, ω_2) only occur in the third, fourth, and fifth equations, and that these equations only include these quantities. We can, therefore, split up the problem of stability into two independent sets of three equations each.

Taking into account the conditions of rectilinear steady motion, viz. $G_{10} = G_{20} = G_{30} = 0$, and the equations (138) in § 109, we find that the disturbance in the plane of symmetry is given by

$$\left.\begin{aligned} \frac{du_1}{dt} - U_{20}\frac{d\theta}{dt} &= g\theta\cos\Theta_0 - U(a_x u_1 + b_x u_2 + f_x\omega_3), \\ \frac{du_2}{dt} + U_{10}\frac{d\theta}{dt} &= -g\theta\sin\Theta_0 - U(a_y u_1 + b_y u_2 + f_y\omega_3), \\ \frac{d^2\theta}{dt^2} &= - U(a_3 u_1 + b_3 u_2 + f_3\omega_3), \end{aligned}\right\} \quad . \quad . \quad (145)$$

and the assumed small motion outside this plane, called the **lateral motion** or **oscillation**, is given by the equations

$$\left.\begin{aligned} &\frac{du_3}{dt} - U_{10}\cos\Theta_0 \,.\, \frac{d\psi}{dt} + U_{20}\frac{d\phi}{dt} + U_{20}\sin\Theta_0 \,.\, \frac{d\psi}{dt} \\ &\qquad = -g\cos\Theta_0 \,.\, \phi - U\left(c_z u_3 + d_z \,.\, \overline{\frac{d\phi}{dt} + \sin\Theta_0 \,.\, \frac{d\psi}{dt}} + e_z\cos\Theta_0 \,.\, \frac{d\psi}{dt}\right), \\ &\frac{d^2\phi}{dt^2} + \sin\Theta_0 \,.\, \frac{d^2\psi}{dt^2} - \frac{F}{A}\cos\Theta_0 \,.\, \frac{d^2\psi}{dt^2} \\ &\qquad = - U\left(c_1 u_3 + d_1 \,.\, \overline{\frac{d\phi}{dt} + \sin\Theta_0 \,.\, \frac{d\psi}{dt}} + e_1\cos\Theta_0 \,.\, \frac{d\psi}{dt}\right), \\ &\cos\Theta_0\frac{d^2\psi}{dt^2} - \frac{F}{B}\frac{d^2\phi}{dt^2} - \frac{F}{B}\sin\Theta_0 \,.\, \frac{d^2\psi}{dt^2} \\ &\qquad = - U\left(c_2 u_3 + d_2 \,.\, \overline{\frac{d\phi}{dt} + \sin\Theta_0 \,.\, \frac{d\psi}{dt}} + e_2\cos\Theta_0 \,.\, \frac{d\psi}{dt}\right), \end{aligned}\right\} (146)$$

The quantities a_x, b_x, etc., c_z, d_z, etc., have been defined (§ 37); also, as in § 87, $U = (U_{10}{}^2 + U_{20}{}^2)^{\frac{1}{2}}$.

It will be seen that the equations (145) are exactly the same as the equations (104) in § 87, on the stability with no lateral motion at all. The *longitudinal stability of the symmetrical aeroplane is, therefore, unaffected by small lateral motions.* The derivatives in (145) are the same numerical quantities as the corresponding derivatives in (104).

For the assumed small lateral motions we have, using the method of § 85 for the variables u_3, ϕ, $d\psi/dt$, the following determinantal equation for λ:

$$\begin{vmatrix} \lambda + c_z, & \overline{\frac{U_{20}}{U} + d_z} \,.\, \lambda + \frac{g}{U}\cos\Theta_0, & \frac{U_{20}}{U}\sin\Theta_0 - \frac{U_{10}}{U}\cos\Theta_0 + \sin\Theta_0 \,.\, d_z + \cos\Theta_0 \,.\, e_z \\ c_1, & \lambda^2 + d_1 \,.\, \lambda \quad , & \overline{\sin\Theta_0 - \frac{F}{A}\cos\Theta_0} \,.\, \lambda + \sin\Theta_0 \,.\, d_1 + \cos\Theta_0 \,.\, e_1 \\ c_2, & -\frac{F}{B}\lambda^2 + d_2 \,.\, \lambda \quad , & \overline{\cos\Theta_0 - \frac{F}{B}\sin\Theta_0} \,.\, \lambda + \sin\Theta_0 \,.\, d_2 + \cos\Theta_0 \,.\, e_2 \end{vmatrix} = 0. \quad (147)$$

The quantities u_3, ϕ, $d\psi/dt$ are expressible as the sums of exponentials $e^{U\lambda_1 t}$, $e^{U\lambda_2 t}$ . . ., each multiplied by (analytically arbitrary) constants. These depend on the four initial conditions, viz. u_{30}, ϕ_0, $(d\phi/dt)_0$, $(d\psi/dt)_0$, the suffix denoting that these quantities are to have their values at $t = 0$.

116. **Indifference with Regard to Azimuth.**—It is to be noted that the solution does not contain the initial value of ψ. The angle Ψ merely denotes the direction of the plane which we choose for the longitudinal motion, with reference to some fixed vertical plane. The direction of this standard plane (unlike those for Θ, Φ) is not determined by any natural criterion, and so we can make it coincide with the actual plane of longitudinal motion, so that we can assume that $\psi_0 = 0$, without any loss of generality. This expresses the obvious physical fact that the symmetrical aeroplane will not, of itself, tend to make its plane of symmetry assume any one vertical position.

117. **Conditions of Lateral Stability.**—The determinantal equation (147) can be simplified by multiplying the third column by λ, subtracting from this the second column multiplied by $\sin\Theta_0$, and taking out the factor $\cos\Theta_0$ in the resulting new third column. We get

$$\begin{vmatrix} \lambda + c_z, & d_z + \frac{U_{20}}{U}.\lambda + \frac{g}{U}\cos\Theta_0, & e_z - \frac{U_{10}}{U}.\lambda - \frac{g}{U}\sin\Theta_0 \\ c_1, & \lambda^2 + d_1\lambda, & -\frac{F}{A}\lambda^2 + e_1\lambda \\ c_2, & -\frac{F}{B}\lambda^2 + d_2\lambda, & \lambda^2 + e_2\lambda \end{vmatrix} = 0. \quad (148)$$

in which we know that a factor λ can be cancelled out. We get an equation

$$A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2 = 0,$$

where A_2, B_2, C_2, D_2, E_2 can be evaluated. The aeroplane is laterally stable if this equation has the real parts of all its solutions negative (or at least not positive). The conditions are that A_2, B_2, C_2, D_2, E_2 shall all be of the same sign, as well as Routh's discriminant

$$H_2 \equiv B_2C_2D_2 - A_2D_2^2 - E_2B_2^2.$$

In the standard case where the steady motion is horizontal with the propeller axis horizontal (normal motion), we have $U_{20}=0$, $\Theta_0=0$, $U_{10}=U$, and the above determinant is slightly simplified.

It must, in any case, be clearly understood that the derivatives in both the longitudinal and lateral conditions refer to the configuration of the body appropriate to the steady motion to which the motion approximates. Thus, *e.g.*, the derivatives in (145, 146) will all involve U_{20} and Θ_0, and will depend on the kind of symmetrical steady motion considered. In practice, we would take the case of normal motion $U_{20} = \Theta_0 = 0$, and we get a range of steady motions for which the stability holds, if it holds for this case.

118. **General Case of Rectilinear Steady Motion.**—The discussion of the stability of the general case of rectilinear steady motion will now present no difficulty, although the algebra will be considerably heavier. We have three velocity components U_{10}, U_{20}, U_{30}; once again $\Omega_{10} = \Omega_{20} = \Omega_{30} = 0$. We put

$$U_1 = U_{10} + u_1,\ U_2 = U_{20} + u_2,\ U_3 = U_{30} + u_3,\ \Omega_1 = \omega_1,\ \Omega_2 = \omega_2,\ \Omega_3 = \omega_3;$$
$$\Theta = \Theta_0 + \theta,\ \Phi = \Phi_0 + \phi,\ \Psi = \Psi_0 + \psi,$$

and ignore squares and products, etc., of the small letters. We now have,

speaking generally, all the thirty-six derivatives, and the relations between ω_1, ω_2, ω_3 and θ, ϕ, ψ are

$$\omega_1 = \frac{d\phi}{dt} + \sin\Theta_0 \,.\, \frac{d\psi}{dt},$$

$$\omega_2 = \sin\Phi_0 \frac{d\theta}{dt} + \cos\Theta_0 \cos\Phi_0 \frac{d\psi}{dt},$$

$$\omega_3 = \cos\Phi_0 \frac{d\theta}{dt} - \cos\Theta_0 \sin\Phi_0 \frac{d\psi}{dt}.$$

It will be found that the stability depends on a determinantal equation for λ, in which this symbol occurs to the eighth power. It is, therefore, necessary to have the real parts of all the eight roots of this equation negative (or not positive). There would be no advantage in giving the algebra in full; the student can easily supply it, and he is recommended to go through the work for its educational value. The important thing to notice is that owing to the lack of symmetry, the longitudinal and lateral motions do not separate out, but remain entangled in all the equations of motion.

At this stage we can also include, if we please, the gyroscopic effects of the propeller and engine, as well as all the derivatives of T. Thus, even for a symmetrical aeroplane, we get the longitudinal and lateral motions entangled, although for practical purposes this fact can be ignored.

119. **General Steady Motion.**—It must not be supposed, however, that the straight-line steady motion is the only type of steady motion possible. Steady motion being defined as one in which the components of translational and rotational velocity remain constant, we have such motion if we can make U_1, U_2, U_3, Ω_1, Ω_2, Ω_3 all constant throughout the motion. It follows from equations (136) that G_1, G_2, G_3 must be constant as well as

$$g\sin\Theta + T - R_1, \quad g\cos\Theta\cos\Phi - R_2, \quad -g\cos\Theta\sin\Phi - R_3.$$

The first three are constant because of the steadiness of the motion; $T - R_1$, R_2, R_3 are constant for the same reason. Hence we must have

$$\sin\Theta, \quad \cos\Theta\cos\Phi, \quad \cos\Theta\sin\Phi$$

constant throughout the motion, if it is to be called steady. We get, then, that in steady motion Θ and Φ must be constant. There is no restriction on Ψ so far.

But in steady motion Ω_1, Ω_2, Ω_3 are constants; putting $d\Theta/dt = 0$, $d\Phi/dt = 0$, in the equations (137), we get

$$\frac{d\Psi}{dt}\sin\Theta, \quad \frac{d\Psi}{dt}\cos\Theta\cos\Phi, \quad -\frac{d\Psi}{dt}\cos\Theta\sin\Phi,$$

all constant throughout the motion. It follows that $d\Psi/dt$ must be a constant. Hence in steady motion we have Θ, Φ, $d\Psi/dt$ constants.

For a particular type of steady motion, *i.e.* given component velocities and angular velocities, we require a certain shape of the body, *i.e.* certain

positions of the elevator, rudder, and ailerons, and also a certain thrust. If any change is made, either in the shape or in the condition of the engine, a different steady motion is appropriate, and in this way (as well as by disturbances in general) the problem of the maintenance of, and approach to, a given state of generalised steady motion arises.

Rectilinear steady flight is a special case in which not only are Θ and Φ constant, but also Ψ, so that $d\Psi/dt$ is zero. This has been already considered. The next case, more general than this, but still specialised, is that of motion in a circle in a horizontal plane.

120. **Circular Flight; Side Slip.**—In order to discuss this as well as the more general steady motion, let us see what components of velocity dx/dt, dy/dt, dz/dt along the space axes Ox, Oy, Oz contribute respectively to the components U_1, U_2, U_3 along the body axes. Referring to Fig. 55, we see that a velocity dx/dt along Ox gives along X_1, Y_1, Z_1 the components

$$\frac{dx}{dt}\cos\Psi,\quad 0,\quad \frac{dx}{dt}\sin\Psi.$$

These give along X_2, Y_2, Z_2 (Fig. 56) the components

$$\frac{dx}{dt}\cos\Psi\cos\Theta_0,\quad -\frac{dx}{dt}\cos\Psi\sin\Theta_0,\quad \frac{dx}{dt}\sin\Psi,$$

so that along X_3, Y_3, Z_3 (*i.e.* X, Y, Z), Fig. 57, we get components

$$\frac{dx}{dt}\cos\Psi\cos\Theta_0,\quad \frac{dx}{dt}(\sin\Psi\sin\Phi_0-\cos\Psi\sin\Theta_0\cos\Phi_0),$$
$$\frac{dx}{dt}(\sin\Psi\cos\Phi_0+\cos\Psi\sin\Theta_0\sin\Phi_0).$$

By similar arguments we find that dy/dt along Oy, and dz/dt along Oz, give components

$$\frac{dy}{dt}\sin\Theta_0,\quad \frac{dy}{dt}\cos\Theta_0\cos\Phi_0,\quad -\frac{dy}{dt}\cos\Theta_0\sin\Phi_0,$$

$$-\frac{dz}{dt}\sin\Psi\cos\Theta_0,\quad \frac{dz}{dt}(\cos\Psi\sin\Phi_0+\sin\Psi\sin\Theta_0\cos\Phi_0),$$
$$\frac{dz}{dt}(\cos\Psi\cos\Phi_0-\sin\Psi\sin\Theta_0\sin\Phi_0).$$

We, therefore, get

$$\left.\begin{aligned}
U_{10} &= \cos\Theta_0\cos\Psi\,.\,\frac{dx}{dt}+\sin\Theta_0\,.\,\frac{dy}{dt}-\cos\Theta_0\sin\Psi\,.\,\frac{dz}{dt},\\
U_{20} &= (\sin\Phi_0\sin\Psi-\sin\Theta_0\cos\Phi_0\cos\Psi)\frac{dx}{dt}+\cos\Theta_0\cos\Phi_0\,.\,\frac{dy}{dt}\\
&\qquad +(\sin\Phi_0\cos\Psi+\sin\Theta_0\cos\Phi_0\sin\Psi)\frac{dz}{dt},\\
U_{30} &= (\cos\Phi_0\sin\Psi+\sin\Theta_0\sin\Phi_0\cos\Psi)\frac{dx}{dt}-\cos\Theta_0\sin\Phi_0\,.\,\frac{dy}{dt}\\
&\qquad +(\cos\Phi_0\cos\Psi-\sin\Theta_0\sin\Phi_0\sin\Psi)\frac{dz}{dt}.
\end{aligned}\right\}\quad(149)$$

Now in steady motion in a horizontal circle we must have $dy/dt = 0$, and, as is made evident in Fig. 59,

$$\frac{dx}{dt} = U \cos \Psi', \quad \frac{dz}{dt} = - U \sin \Psi',$$

where U is the speed in the circle, and Ψ' is the angle between the instantaneous direction of motion and Ox. We, therefore, get

$$\begin{aligned}
U_{10} &= U \cos \Theta_0 \cos (\Psi' - \Psi), \\
U_{20} &= U\{- \sin \Phi_0 \sin (\Psi' - \Psi) - \sin \Theta_0 \cos \Phi_0 \cos (\Psi' - \Psi)\}, \\
U_{30} &= U\{- \cos \Phi_0 \sin (\Psi' - \Psi) + \sin \Theta_0 \sin \Phi_0 \cos (\Psi' - \Psi)\}.
\end{aligned}$$

But U_{10}, U_{20}, U_{30}, Θ_0, Φ_0 are constants, and so is U, which is

$$(U_{10}{}^2 + U_{20}{}^2 + U_{30}{}^2)^{\frac{1}{2}}.$$

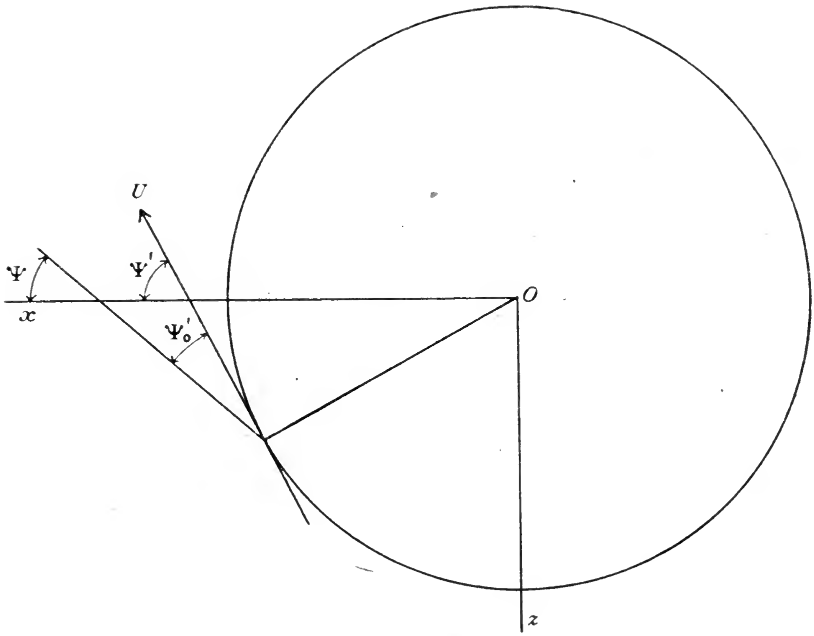

FIG. 59.—Circular Steady Motion. Side-slip.

Hence $\Psi' - \Psi$ is constant throughout the motion, so that, as $d\Psi/dt$ is a constant, so is $d\Psi'/dt$. Let us call each of these angular velocities Ω. Then

$$\Psi = \Omega t + \Psi_0, \quad \Psi' = \Omega t + \Psi_0{}',$$

where Ψ_0, Ψ_0 are the values of Ψ, Ψ' at $t = 0$. We can choose the plane of xy, *i.e.* the original plane for measuring Ψ, in any way we please or find convenient. Let us make Ψ_0 zero; then $\Psi' - \Psi$ is permanently $\Psi_0{}'$, and hence

$$\begin{aligned}
U_{10} &= U \cos \Theta_0 \cos \Psi_0{}', \\
U_{20} &= U(- \sin \Phi_0 \sin \Psi_0{}' - \sin \Theta_0 \cos \Phi_0 \cos \Psi_0{}'), \\
U_{30} &= U(- \cos \Phi_0 \sin \Psi_0{}' + \sin \Theta_0 \sin \Phi_0 \cos \Psi_0{}').
\end{aligned}$$

The vertical plane in which the X axis of the machine lies makes an angle Ψ with the original xy plane (§ 106). Hence Ψ_0' is the angle this vertical plane makes at any moment with the vertical plane containing the direction in which the centre of gravity of the machine is moving at the instant, *i.e.* the angle of yaw. Thus, in general, steady motion in a circle can be performed with the resultant motion inclined to the X axis of the machine, and there is **side slipping**, made evident to the flyer by the relative wind appearing to come from the side. When Ψ_0' is positive, the wind appears to come from the side towards which the turning is directed (*i.e.* from the right in our figure); when Ψ_0' is negative, the wind appears to come from the opposite side (from the left). No side-slipping occurs when Ψ_0' is exactly zero.

121. **Banking.**—Consider first the case of no side-slip. We get

$$U_{10} = U \cos \Theta_0, \quad U_{20} = - U \sin \Theta_0 \cos \Phi_0, \quad U_{30} = U \sin \Theta_0 \sin \Phi_0.$$

The angular velocities are

$$\Omega_{10} = \Omega \sin \Theta_0, \quad \Omega_{20} = \Omega \cos \Theta_0 \cos \Phi_0, \quad \Omega_{30} = - \Omega \cos \Theta_0 \sin \Phi_0.$$

Substituting in the first three equations (136),

$$\left(\text{with } \frac{dU_1}{dt} = \frac{dU_2}{dt} = \frac{dU_3}{dt} = 0\right),$$

we get

$$\left.\begin{aligned} -g \sin \Theta_0 - T_0 + R_{10} &= 0, \\ -g \cos \Theta_0 \cos \Phi_0 + R_{20} &= \Omega U \sin \Phi_0, \\ g \cos \Theta_0 \sin \Phi_0 + R_{30} &= \Omega U \cos \Phi_0. \end{aligned}\right\} \qquad (150)$$

The coefficient ΩU is, of course, the well-known (velocity)^2/(radius of path) or (angular velocity)^2 × (radius of path) which occurs in the uniform motion of a body round a circle.

It is now clear that there must be definite values of Θ_0, Φ_0 for given velocity and radius of the path. The angle Φ_0 is called the *angle of bank*; if no side-slipping is to occur, this must be chosen appropriately, so that for the given shape of the body and circumstances of engine and motion there may be statical equilibrium of the forces acting on the machine and the so-called centrifugal forces.

We may conceive a case in which no banking is necessary, so that the machine has its wings horizontal the whole time, Φ_0 being zero. This would require $R_{30} = \Omega U$, and it can be shown that the size of rudder required for the purpose is quite unwieldy and impracticable.

In actual machines, if the path is only slightly curved, R_{30} is comparatively small, since it is zero if the curvature is zero and the machine is symmetrical. The required angle of bank is, therefore, given approximately by

$$\tan \Phi_0 = \frac{\Omega U}{g \cos \Theta_0} = \frac{U^2}{g\rho \cos \Theta_0}, \qquad (151)$$

where ρ is the radius of the circular path. This shows that the required angle of bank for zero side-slip increases with the curvature of the path. Also Φ_0 has the same sign as Ω, so that to turn to the right the machine must roll to the right in banking, and to turn to the left it must roll to the left.

Further, for a given speed, the second equation (150) gives

$$R_{20} = g \cos \Theta_0 \cos \Phi_0 + \Omega U \sin \Phi_0,$$

so that the lift must be increased by the amount $\Omega U \sin \Phi_0$, which is always positive, since Ω and Φ_0 are of the same sign. The elevator must, therefore, be turned down so as to increase the angle of attack, at the same time as the rudder and ailerons are given their proper positions for the turn. We have, in fact,

$$R_{20} = g \cos \Theta_0 \sec \Phi_0.$$

In the general case, with side-slip defined by Ψ_0', we get, instead of (150), the equations

$$\left.\begin{aligned} -g \sin \Theta_0 - T_0 + R_{10} &= \Omega U \cos \Theta_0 \sin \Psi_0', \\ -g \cos \Theta_0 \cos \Phi_0 + R_{20} &= \Omega U (\sin \Phi_0 \cos \Psi_0' - \sin \Theta_0 \cos \Phi_0 \sin \Psi_0'), \\ g \cos \Theta_0 \sin \Phi_0 + R_{30} &= \Omega U (\cos \Phi_0 \cos \Psi_0' + \sin \Theta_0 \sin \Phi_0 \sin \Psi_0'). \end{aligned}\right\} \quad . \quad . \quad (152)$$

It is important to notice that the machine will fly in a circle so long as the conditions for circular flight are maintained. To make it turn and then fly in a straight line, the controls must be released. In the case of climbing or descending in longitudinal flight, the controls must be kept in the new positions so long as the climb or descent is maintained.

122. **Stability of Circular Flight.**—The study of the stability of circular flight, *i.e.* to find whether a machine with its shape and engine arranged for a certain steady circular motion in a horizontal plane will tend to assume this motion if at any moment, by some cause, its motion happens to be slightly divergent, is carried out in exactly the same manner as the stability problems already discussed. There are certain definite values of

$$U_{10},\ U_{20},\ U_{30},\ \Omega_{10},\ \Omega_{20},\ \Omega_{30},\ \Theta_0,\ \Phi_0,\ \left(\frac{d\Psi}{dt}\right)_0,\ T_0,\ R_{10},\ R_{20},\ R_{30},\ G_{10},\ G_{20},\ G_{30}.$$

We put

$$U_1 = U_{10} + u_1,\ \ U_2 = U_{20} + u_2,\ \ U_3 = U_{30} + u_3,\ \ \Omega_1 = \Omega_{10} + \omega_1,\ \ \Omega_2 = \Omega_{20} + \omega_2,$$

$$\Omega_3 = \Omega_{30} + \omega_3,\ \ \Theta = \Theta_0 + \theta,\ \ \Phi = \Phi_0 + \phi,\ \ \frac{d\Psi}{dt} = \left(\frac{d\Psi}{dt}\right)_0 + \frac{d\psi}{dt},$$

$$R_1 - T = R_{10} - T_0 + U(a_x u_1 + b_x u_2 + c_x u_3 + d_x \omega_1 + e_x \omega_2 + f_x \omega_3),$$

and similarly for R_2, R_3, G_1, G_2, G_3, where $U = (U_{10}{}^2 + U_{20}{}^2 + U_{30}{}^2)^{\frac{1}{2}}$; the small letters $u_1, u_2, \ldots$ denote small quantities, which, stability assumed, will remain small all through the motion, so that their squares, products, etc., may be neglected. If we substitute in the equations of motion (136), using the relations given by (137), we get six simultaneous linear differential equations for the six quantities u_1, u_2, u_3, θ, ϕ, $d\psi/dt$. Using the method of § 81, we get a determinantal equation in λ, which, when worked out, gives an octic, *i.e.* an equation of the eighth degree. For stability, all the eight roots must have their real parts negative. There is, of course, no separation of the longitudinal and lateral motions, since there is no symmetry in the machine or in the steady motion.

123. **Helical Steady Motion.**—We now proceed to the examination of the most general steady motion.

It has been shown that in steady motion we must have Θ, Φ, $d\Psi/dt$ constant (§ 119). Also U_1, U_2, U_3, Ω_1, Ω_2, Ω_3 are constants. Consider now the equations (149), which give U_{10}, U_{20}, U_{30} in terms of dx/dt, dy/dt,

dz/dt. Let us solve them so as to get dx/dt, dy/dt, dz/dt in terms of U_{10}, U_{20}, U_{30}. (We can also obtain these results by means of geometrical arguments from Figs. 55, 56, 57, taken backwards.) We get

$$\left.\begin{aligned}\frac{dx}{dt} &= \cos\Theta_0\cos\Psi\,.\,U_{10}+(\sin\Phi_0\sin\Psi-\sin\Theta_0\cos\Phi_0\cos\Psi)\,U_{20}\\ &\qquad+(\cos\Phi_0\sin\Psi+\sin\Theta_0\sin\Phi_0\cos\Psi)\,U_{30},\\ \frac{dy}{dt} &= \sin\Theta_0\,.\,U_{10}+\cos\Theta_0\cos\Phi_0\,.\,U_{20}-\cos\Theta_0\sin\Phi_0\,.\,U_{30},\\ \frac{dz}{dt} &= -\cos\Theta_0\sin\Psi\,.\,U_{10}+(\sin\Phi_0\cos\Psi+\sin\Theta_0\cos\Phi_0\sin\Psi)\,U_{20}\\ &\qquad+(\cos\Phi_0\cos\Psi-\sin\Theta_0\sin\Phi_0\sin\Psi)\,U_{30}.\end{aligned}\right\}\quad(15\!:$$

Thus dy/dt is independent of Ψ and, therefore, constant throughout the motion. The vertical fall or rise is thus uniform. But dx/dt, dz/dt are not constant.

If, however, we calculate the value of $(dx/dt)^2+(dz/dt)^2$, we find it to be independent of Ψ; it is, in fact,

$$\begin{aligned}\cos^2\Theta_0\,.\,U_{10}^2&+(\sin^2\Phi_0+\sin^2\Theta_0\cos^2\Phi_0)\,U_{20}^2+(\cos^2\Phi_0+\sin^2\Theta_0\sin^2\Phi_0)\,U_{30}^2\\ &+2\cos^2\Theta_0\sin\Phi_0\cos\Phi_0\,.\,U_{20}U_{30}+2\sin\Theta_0\cos\Theta_0\sin\Phi_0\,.\,U_{30}U_{10}\\ &\qquad-2\sin\Theta_0\cos\Theta_0\cos\Phi_0\,.\,U_{10}U_{20}.\end{aligned}$$

Hence the horizontal speed of the centre of gravity of the machine is constant. We easily deduce

$$\sin\Psi\frac{dx}{dt}+\cos\Psi\frac{dz}{dt}=\sin\Phi_0\,.\,U_{20}+\cos\Phi_0\,.\,U_{30},$$

$$\cos\Psi\frac{dx}{dt}-\sin\Psi\frac{dz}{dt}=\cos\Theta_0\,.\,U_{10}-\sin\Theta_0\cos\Phi_0\,.\,U_{20}+\sin\Theta_0\sin\Phi_0\,.\,U_{30},$$

which are also independent of Ψ and, therefore, constant. Let us, by analogy with § 120, write

$$\frac{dx}{dt}=\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}\cos\Psi',\quad \frac{dz}{dt}=-\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}\sin\Psi',$$

where Ψ' is the angle between the vertical plane containing the instantaneous direction of motion and the vertical plane of xy. We find

$$\left.\begin{aligned}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}\sin(\Psi'-\Psi)&=-\sin\Phi_0\,.\,U_{20}-\cos\Phi_0\,.\,U_{30},\\ \left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}\cos(\Psi'-\Psi)&=\cos\Theta_0\,.\,U_{10}-\sin\Theta_0\cos\Phi_0\,.\,U_{20}+\sin\Theta_0\sin\Phi_0\,.\,U_{30}.\end{aligned}\right\}\quad(154$$

It follows that in the most general steady motion, as in the case of circular motion, we have $\Psi'-\Psi$ constant, and we can call this constant Ψ_0' by proper choice of the original plane of xy. If we designate the constant value of $d\Psi/dt$ by Ω, we have

$$\Psi'=\Omega t+\Psi_0'.$$

It is, therefore, clear that the horizontal projection of the motion of the centre of gravity in the most general steady motion is uniform motion in a circle. The general steady motion is thus uniform circular motion with a uniform fall or rise; in other words, the centre of gravity describes with uniform speed a helix on a vertical circular cylinder.

To visualise this motion, imagine a vertical circular cylinder with one

screw thread round it, of constant pitch. The centre of gravity runs up or down with constant speed. In Fig. 60 the arrows on the helix show a case where the aeroplane is descending with a positive rotation (to the right). The other arrows show a case where the aeroplane is climbing with a negative rotation (to the left). The equations for steady helical flight, corresponding to (152) in circular flight, are left as an exercise to the student. They will be found to be exactly the same as in circular flight, U being replaced by $\{(dx/dt)^2 + (dz/dt)^2\}^{\frac{1}{2}}$.

124. The importance of the generalised steady motion for practical flight cannot be overestimated. If a flyer wishes to rise to a considerable

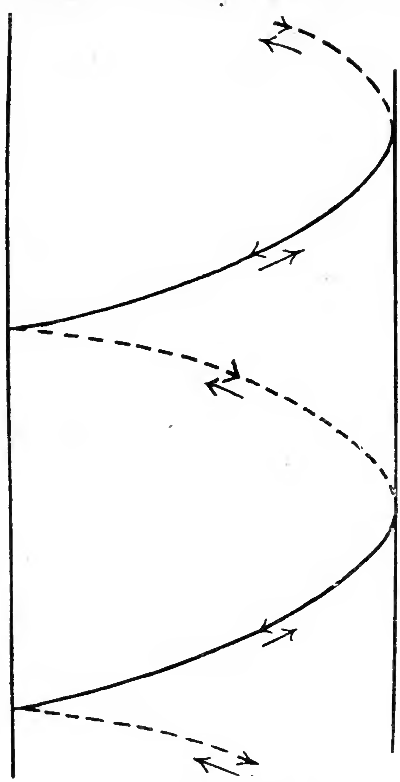

FIG. 60.—Helical Steady Motion.

height, then, since climbing is only a gradual process, the angle of climb being a moderate number of degrees, the result of a straight-line climb would be to take him away a large distance from his starting point. By climbing in a helical path he can rise to any desired height without going far away from his starting point. The same applies to descending after a flight, or hovering over a given locality during flight, when the natural thing to do, since it is not possible to stop in the air, is to describe a circular path.

In general, the engine is required for a helical path, especially when rising, and even when going down in a more or less flat helix. When the thrust is reduced to zero and the machine is in a given condition as to elevator, rudder, and ailerons, the appropriate steady motion is, in

general, a helical glide. The gliding angle is $\sin^{-1}\left(\frac{dy}{dt}/U\right)$, where U is the resultant speed $(U_{10}^2 + U_{20}^2 + U_{30}^2)^{\frac{1}{2}}$.

In the helical steady motion there is a side-slip, defined by the angle Ψ_0', which is the angle between the (vertical) tangent plane to the cylinder and the contemporaneous vertical plane through the X axis in the body. To obtain zero side-slip we must have, by (154),

$$\sin \Phi_0 \,.\, U_{20} + \cos \Phi_0 \,.\, U_{30} = 0,$$

which presupposes the appropriate angle of bank.

The discussion of the stability of a helical steady motion is in every respect similar to that for the case of circular motion. An octic is obtained for λ, and the conditions of stability are that the eight roots shall all have their real parts negative.

125. **Rectilinear and Circular Steady Flight as Special Cases.**—We can sum up the facts of the problem of steady motion as follows.

If we have a given machine with given state of the controls and with a given condition of the engine and propeller, there is a definite appropriate steady motion. This is, in general, helical, and from the given data we can imagine that we can find the helix (*i.e.* the radius of the vertical cylinder on which it is described, and the angle of descent or ascent on this cylinder), the speed on the helix, and the orientation of the machine relatively to the helix.

Two special cases can arise. The radius of the cylinder on which the helix is described may become infinite, or $\Omega = 0$, in § 123. In this case we get a rectilinear steady motion, which may be along any straight line, and with the machine in any orientation relatively to this straight line. Thus we can get non-symmetrical rectilinear steady motion, or symmetrical linear steady motion in the vertical position of the plane of symmetry of the machine, but *in any direction up or down*, or, more particularly, a glide or horizontal steady motion, or any other such special form.

If the angle of descent or ascent of the helix on the cylinder is zero, the helix reduces to a circle in a horizontal plane. This special case, therefore, gives us the case of steady circular flight. The analytical condition is that $dy/dt = 0$ in § 123. This circular motion may be at any speed and with any curvature of the path. In particular it may happen that the curvature of the circular path is zero, and the circle becomes a straight line. We then have horizontal rectilinear motion.

We thus have the following scheme :

Helical Steady Motion (General)

Special Case, $\Omega = 0$ Rectilinear Steady Motion (General)	*Special Case*, $dy/dt = 0$ Circular Steady Motion
Special Case	*Special Case*, $\Omega = 0$
Symmetrical	
Special Case	

Horizontal Symmetrical Steady Motion

It appears to be no easy task to discuss, in a general manner, the conditions that one or other form of steady motion shall arise. The practical flyer relies upon his experience of the machine he is using.

126. **More General Problem; Spiral Nose Dive.**—We can, by analogy with § 89, construct an approximate theory for a non-symmetrical dive in the form of a *spiral nose dive.* Let the propeller thrust be zero. The general case admits of similar treatment.

The machine with no propeller thrust has no definite set of body axes.

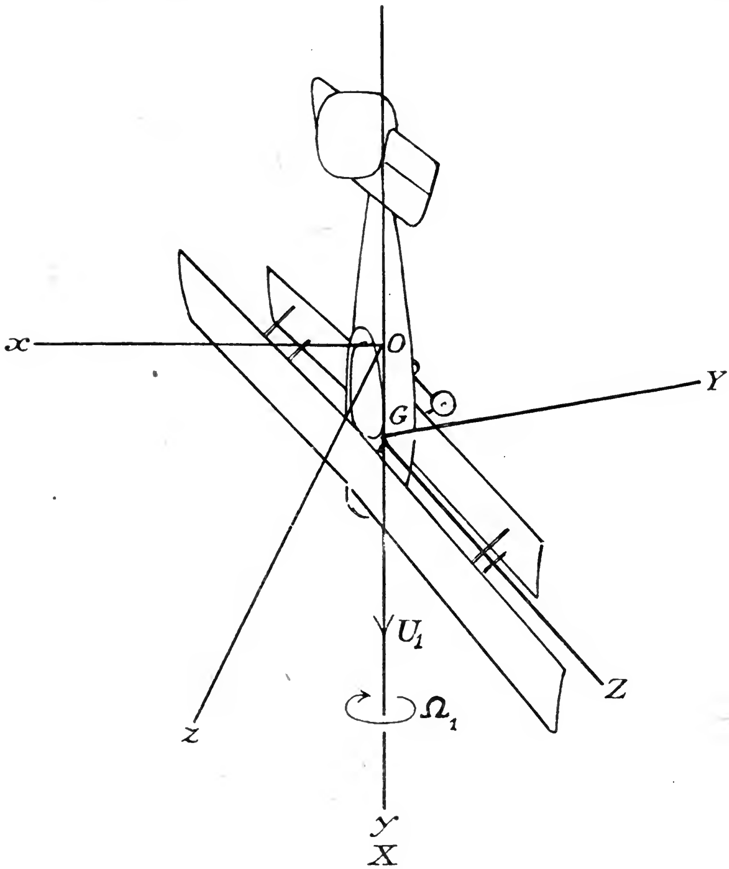

FIG. 61.—Spiral Nose Dive of Aeroplane.

(It will be remembered that we have all through our work taken the X axis along the propeller axis, which was assumed to pass through the centre of gravity.) Let us, then, leave the axes undefined at present, but more or less in the same positions as the body axes used so far. The equations of motion (136) refer, of course, to any set of body axes through the centre of gravity so long as they are right-handed and defined by successive rotations Ψ, Θ, Φ from the standard axes in space Ox, y, z.

Let us try to construct the motion in a spiral dive. As an approximation we can write $U_2 = U_3 = \Omega_2 = \Omega_3 = 0$, it being supposed that the effect of the dive is to make the X axis in the body assume a vertical

position. To obtain $\Omega_2 = \Omega_3 = 0$, we suppose in the equations (137) that $\Psi = 0$, $\Theta = \pi/2$, so that $\Omega_1 = d\Phi/dt$.

If we refer to Figs. 55, 56, and 57 we see that the machine is now pointing downwards, and $d\Phi/dt$ represents the rate at which the plane of the wings rotates round the vertical. The equations of motion are now

$$\frac{dU_1}{dt} = g - R_1, \quad R_2 = 0, \quad R_3 = 0;$$

$$\frac{d^2\Phi}{dt^2} = -G_1, \quad -\frac{F}{B}\frac{d^2\Phi}{dt^2} + \frac{E}{B}\left(\frac{d\Phi}{dt}\right)^2 = -G_2, \quad -\frac{E}{C}\frac{d^2\Phi}{dt^2} - \frac{F}{C}\left(\frac{d\Phi}{dt}\right)^2 = -G_3.$$

It is not difficult to see that R_2 and R_3 will be small if the axes are correctly chosen, and, in fact, G_2 and G_3 will also be small. R_1 will be more or less proportional to U_1^2, and since U_1 is now taken downwards, we see that the fall is practically that of a parachute, as explained in the symmetrical nose dive. G_1 will be considerable, since we have the wings

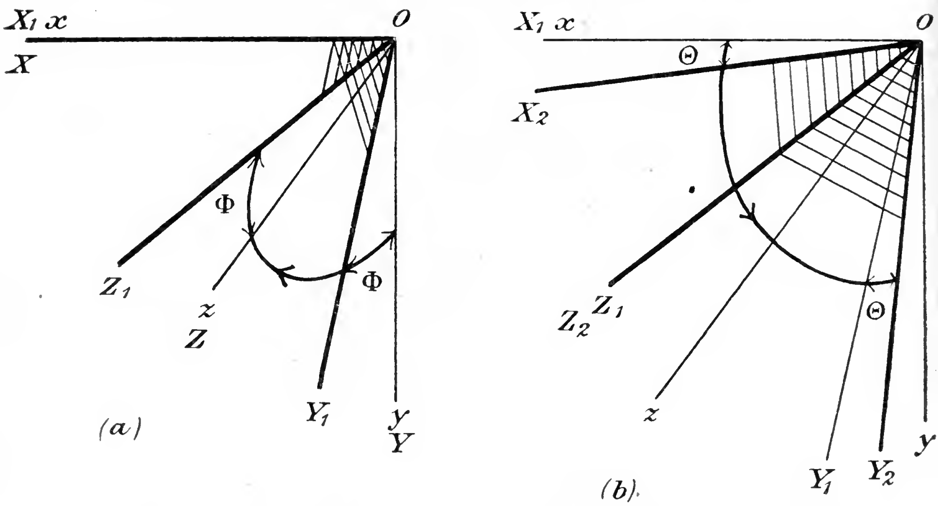

FIG. 62.—Three-Dimensional Motion of Lamina with Simplified Law of Air-Resistance. Space Axes and Body Axes: (a) Roll (Φ); (b) Pitch (Θ).

rotating about an axis lying in them, and perpendicular to their span, and the air will clearly oppose the rotation. The differential equation $d^2\Phi/dt^2 = -G_1$ will involve U_1 and $d\Phi/dt$; similarly, the equation $dU_1/dt = g - R_1$ will involve these quantities. We can thus say that the spiral dive is determined by these two equations.

So long as the axes are in the neighbourhood of the body axes used in the whole of the present work, we can take E and F to be small compared to B and C. Thus the remaining equations are of no significance in our approximate work.

The method of initial motions can be used in three-dimensional aeroplane investigations, the method being similar to that in § 90.

127. **Periodic Solutions.**—We can again apply the idea of periodicity as suggested in the longitudinal problem, Chapter III., §§ 91–92. To illustrate this, we take an extension of the problem there discussed by considering the motion in three dimensions.

The moments of inertia of the body to which the lamina is attached are supposed to be so large that the air pressures produce no change in the angular motion. We shall then assume that the lamina has a constant angular velocity about an axis fixed in the body and in space.

In § 91 we took the axis of rotation to be the Z axis in the body. Since this axis is to remain fixed in direction, we must choose such angular specification in space as will make it convenient to maintain this constancy of direction. In § 106, Figs. 55–57, we took the Ψ, Θ, Φ rotations from the space axes to the body axes in the order indicated. In our problem Ψ does not enter at all, while the variation in Θ should represent the rotation of the body. We must, therefore, carry out the angular displacements in the order (Ψ), Φ, Θ. This is shown in Fig. 62 (a), (b). In Fig. 62 (a) we start off with axes OX, Y, Z coincident with Ox, y, z. We then perform a rotation Φ about the axis OX so that we get a new set of axes OX_1, Y_1, Z_1, such that OX_1 coincides with OX, but OY_1, OZ_1 are turned

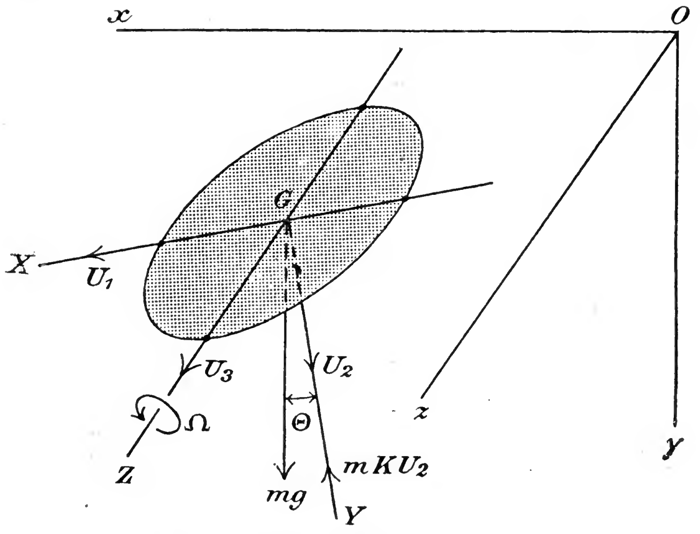

FIG. 63.—Lamina with Simplified Law of Air-Resistance. Axis of Rotation Horizontal and in Plane of Lamina.

in the plane of OYZ through an angle Φ from OY to OZ. In Fig. 62 (b) we perform an angular displacement Θ about the axis OZ_1, so that of the new axes OX_2, Y_2, Z_2, OZ_2 is coincident with OZ_1, whilst OX_2, OY_2 are rotated in the plane OX_1, OY_1 through an angle Θ from OX_1 to OY_1. We do not consider any Ψ displacement, as the azimuth is of no importance. We can now keep Φ constant, and the variation in Θ will represent the angular motion of the lamina and the body to which it is attached.

We readily obtain for the angular velocities of the body

$$\Omega_1 = \Omega_2 = 0, \quad \Omega_3 = \frac{d\Theta}{dt} = \Omega.$$

The gravity components are

$$g \sin \Theta \cos \Phi, \quad g \cos \Theta \cos \Phi, \quad -g \sin \Phi.$$

The equations of motion are the first three in (136) with the new gravity components, since the last three now have no meaning. The air

resistance is taken to be normal to the lamina and proportional to the normal velocity of the centre, where is also the centre of gravity.

Several special cases exist. We shall consider one or two before approaching the general case.

128. **Axis of Rotation Horizontal and in the Plane of the Lamina.**—The equations of motion (136) now become, since $\Phi = 0$ (Fig. 63),

$$\frac{dU_1}{dt} - \Omega U_2 = g \sin \Theta, \quad \frac{dU_2}{dt} + \Omega U_1 = g \cos \Theta - KU_2, \quad \frac{dU_3}{dt} = 0, \quad . \quad . \quad (155)$$

where KU_2 is the air pressure per unit mass with normal velocity U_2. It follows at once that in this case the motion is that given in § 91, with an added constant horizontal velocity as shown by the third equation (155).

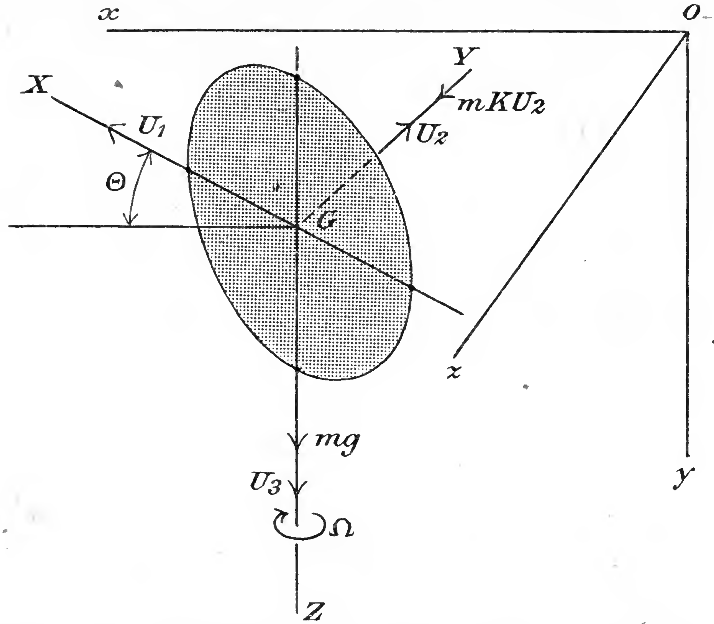

FIG. 64.—Lamina with Simplified Law of Air-Resistance. Axis of Rotation Vertical and in Plane of Lamina.

Hence, after a time, when the initial conditions, *i.e.* the values U_{10}, U_{20}, U_{30} of the velocity components at $t = 0$, have been wiped out, the centre of gravity moves on the average along a straight line whose direction cosines, referred to the fixed axes Ox, y, z, are proportional to $-g/2\Omega$, $2g/K$, U_{30}. The centre of gravity oscillates about this line, and the lamina, meanwhile, rotates about the horizontal axis. The motion can be visualised by supposing the diagram in Figs. 43–44 to move towards the observer with velocity U_{30} (away from the observer if U_{30} is negative), at the same time as the motion in the diagrams is described.

129. **Axis of Rotation Vertical and in the Plane of the Lamina.**—We now take $\Phi = -90°$, and the equations of motion are

$$\frac{dU_1}{dt} - \Omega U_2 = 0, \quad \frac{dU_2}{dt} + \Omega U_1 = -KU_2, \quad \frac{dU_3}{dt} = g \quad . \quad . \quad . \quad . \quad (156)$$

Hence the vertical fall is like that of a particle *in vacuo*, whilst the horizontal motion is given by

$$\frac{dU_1}{d\Theta} - U_2 = 0, \quad \frac{dU_2}{d\Theta} + U_1 = -\frac{K}{\Omega} U_2,$$

where $\Theta = \Omega t$, and is, in fact, the angle through which the lamina has turned from some fixed vertical plane (see Fig. 64). The equations give

$$\frac{d^2U_1}{d\Theta^2} + \frac{K}{\Omega}\frac{dU_1}{d\Theta} + U_1 = 0.$$

If we choose the time origin such that at $t = 0$ (or $\Theta = 0$) the velocity components U_{10}, U_{20} are equal to one another, we find (if $K < 2\Omega$)

$$\left.\begin{aligned} U_1 &= U_{10} e^{-\frac{K}{2\Omega}\Theta}\left\{\left(\frac{2\Omega + K}{2\Omega - K}\right)^{\frac{1}{2}} \sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta + \cos\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta\right\}, \\ U_2 &= U_{10} e^{-\frac{K}{2\Omega}\Theta}\left\{-\left(\frac{2\Omega + K}{2\Omega - K}\right)^{\frac{1}{2}} \sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta + \cos\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta\right\}, \end{aligned}\right\} \quad . \quad (157)$$

a remarkably simple form of the result. (Similar results can be obtained for $K =$ or $> 2\Omega$.) We can readily obtain the horizontal velocities referred to the space axes

$$\left.\begin{aligned} \frac{dx}{dt} &= U_{10} e^{-\frac{K}{2\Omega}\Theta}\left\{\left(\frac{2\Omega + K}{2\Omega - K}\right)^{\frac{1}{2}} \sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta \,.\, (\cos\Theta + \sin\Theta)\right. \\ &\qquad \left. - \cos\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta \,.\, (\sin\Theta - \cos\Theta)\right\}, \\ \frac{dz}{dt} &= U_{10} e^{-\frac{K}{2\Omega}\Theta}\left\{\left(\frac{2\Omega + K}{2\Omega - K}\right)^{\frac{1}{2}} \sin\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta \,.\, (\cos\Theta - \sin\Theta)\right. \\ &\qquad \left. - \cos\left(1 - \frac{K^2}{4\Omega^2}\right)^{\frac{1}{2}}\Theta \,.\, (\cos\Theta + \sin\Theta)\right\}. \end{aligned}\right\} (158)$$

The horizontal motion is thus along a gradually contracting spiral, and ultimately the body falls vertically with a spin about the vertical axis through the centre of gravity. We have, in fact, the case of a **spiral nose dive.**

130. **Axis of Rotation in the Plane of the Lamina, but not Vertical or Horizontal.**—This general case (but the axis of rotation in the plane of the lamina) is given by the equations of motion

$$\left.\begin{aligned} \frac{dU_1}{dt} - \Omega U_2 &= g \sin\Theta \cos\Phi, \\ \frac{dU_2}{dt} + \Omega U_1 &= g \cos\Theta \cos\Phi - KU_2, \\ \frac{dU_3}{dt} &= -g \sin\Phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (159)$$

The motion parallel to the XY plane is now exactly as in the case of § 128, but with $g \cos\Phi$ substituted for g. The motion parallel to the axis of rotation Z is that of a particle under a constant acceleration $-g \sin\Phi$. We thus get parallel to the XY plane (whose direction is fixed in space) a motion like that discussed in the two-dimensional problem; perpendicular to this plane the motion is uniformly accelerated downwards. The ultimate motion is, therefore, the periodic motion in Figs. 43–44 (with $g \cos\Phi$ instead of g) uniformly accelerated normally to the diagrams, the diagrams being held so that their normals make a downward angle Φ with the horizontal. This motion is similar to what has been called the **sideways roll.**

131. **Axis of Rotation Normal to the Lamina.**—This case, illustrated in Fig. 65, will present no difficulty to the student. He will easily find that the motion in the plane of the lamina is, in general, parabolic, whilst normally to the lamina the motion is like that of a parachute.

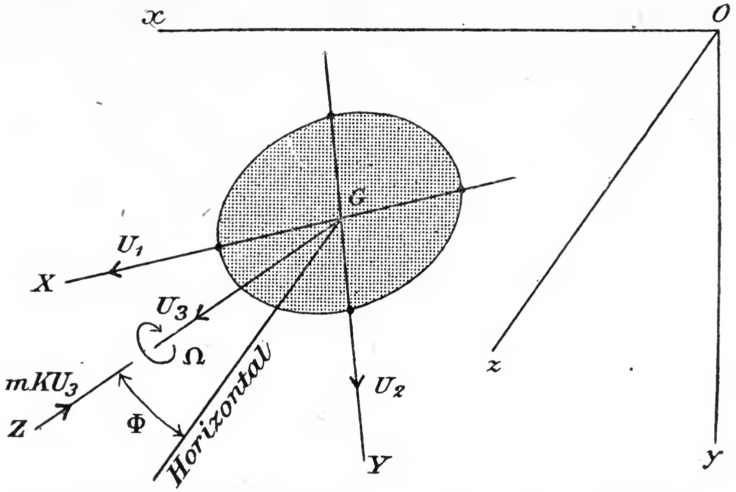

FIG. 65.—Lamina with Simplified Law of Air-Resistance. Axis of Rotation Normal to Lamina.

132. **General Case.**—We now proceed to the general case where the axis of rotation is inclined to the lamina. Let it make an angle α with it. Calling the axis of rotation the Z axis, let the perpendicular to

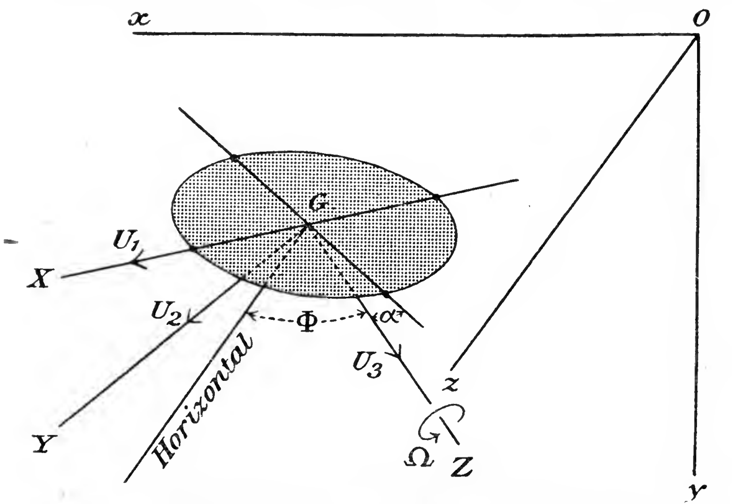

FIG. 66.—Lamina with Simplified Law of Air-Resistance. General Case.

it in the plane of the lamina be the X axis, and choose the Y axis to be perpendicular to both, in such a way that the Y, Z axes are both on the same side of the lamina, and the three axes form a right-handed system, Fig. 66.

The normal velocity component is

$$U_2 \cos a + U_3 \sin a.$$

Hence the normal air pressure is $K(U_2 \cos a + U_3 \sin a)$ per unit mass. This gives components along the Y, Z axes

$$K \cos a\,(U_2 \cos a + U_3 \sin a), \quad K \sin a\,(U_2 \cos a + U_3 \sin a).$$

The equations of motion for the centre of gravity are, therefore,

$$\left.\begin{aligned} \frac{dU_1}{dt} - \Omega U_2 &= g \sin \Theta \cos \Phi, \\ \frac{dU_2}{dt} + \Omega U_1 &= g \cos \Theta \cos \Phi - Km\,(mU_2 + nU_3), \\ \frac{dU_3}{dt} &= -g \sin \Phi - Kn\,(mU_2 + nU_3), \end{aligned}\right\} \quad . \quad . \quad . \quad (160)$$

where we have, for brevity, written $m = \cos a$, $n = \sin a$, so that $m^2 + n^2 = 1$. Since $d\Theta/dt = \Omega$, we get

$$\left.\begin{aligned} \frac{dU_1}{d\Theta} - U_2 &= \frac{g}{\Omega} \sin \Theta \cos \Phi, \\ \frac{dU_2}{d\Theta} + U_1 &= \frac{g}{\Omega} \cos \Theta \cos \Phi - \frac{K}{\Omega} m(mU_2 + nU_3), \\ \frac{dU_3}{d\Theta} &= -\frac{g}{\Omega} \sin \Phi - \frac{K}{\Omega} n(mU_2 + nU_3), \end{aligned}\right\} \quad . \quad . \quad . \quad (161)$$

in which, we remember, Φ is a constant angle. It is easy to eliminate U_1, U_3, and we get for U_2 the equation

$$\frac{d^3U_2}{d\Theta^3} + \frac{K}{\Omega}\frac{d^2U_2}{d\Theta^2} + \frac{dU_2}{d\Theta} + \frac{Kn^2}{\Omega} U_2 = -\frac{2g}{\Omega} \cos \Phi \left(\cos \Theta + \frac{Kn^2}{\Omega} \sin \Theta\right). \quad . \quad (162)$$

If U_2 is found from this equation, U_1, U_3 can be found from (161).

133. The equation (162) is a linear differential equation with constant coefficients. The complementary function is

$$A_1 e^{\lambda_1 \Theta} + A_2 e^{\lambda_2 \Theta} + A_3 e^{\lambda_3 \Theta},$$

where λ_1, λ_2, λ_3 are the solutions of the algebraic equation

$$\lambda^3 + \frac{K}{\Omega}\lambda^2 + \lambda + \frac{Kn^2}{\Omega} = 0.$$

By the conditions given in § 82, we see that the real parts of the roots are all negative. It is, therefore, certain that after some time the complementary function can be left out altogether. (Even the case of equal roots does not really cause any difficulty.) Now the complementary function is the part of U_2 that contains the initial conditions, *i.e.* U_{20}, the initial value of U_2. Hence, after a time, U_{20} disappears from the solution. The ultimate value of U_2 is a Particular Integral of the equation (162), and this is readily seen to be

$$U_2 = \frac{2g \cos \Phi}{Km^2} \cos \Theta + \frac{2g \cos \Phi \, . \, n^2}{m^2 \Omega} \sin \Theta \, . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

From the third equation (161) we find for U_3 the general value

$$-\frac{g}{Kn^2}\sin\Phi-\frac{Kmn}{\Omega}e^{-\frac{Kn^2}{\Omega}\Theta}\int U_2 e^{\frac{Kn^2}{\Omega}\Theta}\,d\Theta.$$

We again see that the part depending on the initial conditions disappears after some time. We find that ultimately

$$\left.\begin{aligned}U_3&=-\frac{g}{Kn^2}\sin\Phi-\frac{2\,ng\cos\Phi}{m\Omega}\sin\Theta,\\ \text{and also}\qquad U_1&=\frac{mg\sin\Phi}{n\Omega}+\frac{2g\cos\Phi}{Km^2}\sin\Theta-\frac{(1+n^2)\,g\cos\Phi}{m^2\Omega}\cos\Theta.\end{aligned}\right\}\quad . \quad . \quad (164)$$

The ultimate velocity components U_1, U_2, U_3 are, therefore, periodic, since Φ is constant and Θ is Ωt. To appreciate the motion more clearly, we write down the velocity components of the centre of gravity referred to axes fixed in space, namely, Ox, y, z. Calling these dx/dt, dy/dt, dz/dt, we have

$$\left.\begin{aligned}\frac{dx}{dt}&=U_1\cos\Theta-U_2\sin\Theta,\\ \frac{dy}{dt}&=(U_1\sin\Theta+U_2\cos\Theta)\cos\Phi-U_3\sin\Phi,\\ \frac{dz}{dt}&=(U_1\sin\Theta+U_2\cos\Theta)\sin\Phi+U_3\cos\Phi,\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (165)$$

so that

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{2n^2g\cos\Phi}{m^2\Omega}+\frac{mg\sin\Phi}{n\Omega}\cos\Theta-\frac{g\cos\Phi}{2\Omega}(1+\cos 2\Theta),\\ \frac{dy}{dt}&=\frac{g(2n^2\cos^2\Phi+m^2\sin^2\Phi)}{Km^2n^2}+\frac{(2n^2+m^2)g\sin\Phi\cos\Phi}{mn\Omega}\sin\Theta-\frac{g\cos^2\Phi}{2\Omega}\sin 2\Theta,\\ \frac{dz}{dt}&=\frac{(2n^2-m^2)g\sin\Phi\cos\Phi}{Km^2n^2}-\frac{g(2n^2\cos^2\Phi-m^2\sin^2\Phi)}{mn\Omega}\sin\Theta-\frac{g\sin\Phi\cos\Phi}{2\Omega}\sin 2\Theta.\end{aligned}\right\}\ (16$$

These are once more periodic functions. The average motion is

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{2n^2g\cos\Phi}{m^2\Omega}-\frac{g\cos\Phi}{2\Omega}=-\frac{(1+3n^2)g\cos\Phi}{2m^2\Omega},\\ \frac{dy}{dt}&=\frac{(2n^2\cos^2\Phi+m^2\sin^2\Phi)\,g}{Km^2n^2},\\ \frac{dz}{dt}&=\frac{(2n^2-m^2)\,g\sin\Phi\cos\Phi}{Km^2n^2}.\end{aligned}\right\}\quad . \quad . \quad (167)$$

The average path is along a straight line whose direction is defined by these component velocities. For positive Ω, *i.e.* rotation from X to Y, and $2n^2>m^2$, *i.e.* $a>\tan^{-1}(1/\sqrt{2})$, the directions of these components are indicated in Fig. 67. The component dx/dt depends on the rotation, the components dy/dt, dz/dt on K. (See § 92.)

The important thing to notice is that the average rate of vertical fall is

$$\frac{g}{K}\left(\frac{\sin^2\Phi}{\sin^2 a}+2\frac{\cos^2\Phi}{\cos^2 a}\right),\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

whilst the rate of fall of the lamina as a parachute would be g/K.

134. **Axis of Rotation Vertical.** — Using $\Phi = 90°$, we get $dx/dt = dz/dt = 0$ for the average motion given by the equations (167), so that the ultimate motion is about a vertical straight line. In this case the average rate of vertical fall is $\frac{g}{K}\operatorname{cosec}^2 a$. The velocity components in space are

$$\frac{dx}{dt} = \frac{g\cot a}{\Omega}\cos\Theta, \quad \frac{dy}{dt} = \frac{g}{K}\operatorname{cosec}^2 a, \quad \frac{dz}{dt} = \frac{g\cot a}{\Omega}\sin\Theta,$$

and the velocity components parallel to the body axes are

$$U_1 = \frac{g}{\Omega}\cot a, \quad U_2 = 0, \quad U_3 = -\frac{g}{K}\operatorname{cosec}^2 a.$$

The centre of gravity of the lamina thus moves along a helix on a vertical circular cylinder, with constant speed, of which the horizontal component is $g\cot a/\Omega$, and the vertical component $g\operatorname{cosec}^2 a/K$; the angle of the helix is, therefore, $\tan^{-1}(\Omega/K\sin a\cos a)$ with the horizontal, so that it decreases, and the helix is flatter, as the resistance becomes more powerful, but increases as the rotation increases. It is to be noted that to get the flattest helix we must use $a = 45°$; also that the horizontal motion is independent of the resistance coefficient, and the vertical motion independent of the rotation.

135. **Axis of Rotation Horizontal.**—A very interesting case arises when the axis of rotation is horizontal so that $\Phi = 0$. The average motion in space is now given by

$$\frac{dx}{dt} = -\frac{g}{2\Omega}(1 + 4\tan^2 a), \quad \frac{dy}{dt} = \frac{2g}{k}\sec^2 a, \quad \frac{dz}{dt} = 0.$$

The actual velocities in space are

$$\left.\begin{aligned}
\frac{dx}{dt} &= -\frac{g}{2\Omega}(1 + 4\tan^2 a) - \frac{g}{2\Omega}\cos 2\Theta,\\
\frac{dy}{dt} &= \frac{2g\sec^2 a}{K} - \frac{g}{2\Omega}\sin 2\Theta,\\
\frac{dz}{dt} &= -\frac{2g\tan a}{\Omega}\sin\Theta.
\end{aligned}\right\} \quad \ldots\ldots \quad (169)$$

Let us imagine a point to run down the average straight-line path with the correct average velocity starting at coincidence with the centre of gravity. If x', y', z' are the co-ordinates of the centre of gravity relatively to the point thus defined, with axes parallel to Ox, y, z, we have

$$x' = -\frac{g}{2\Omega}\int_0^t \cos 2\Theta\,.\,dt, \quad y' = -\frac{g}{2\Omega}\int_0^t \sin 2\Theta\,.\,dt, \quad z' = -\frac{2g\tan a}{\Omega}\int_0^t \sin\Theta\,.\,dt;$$

i.e.

$$x' = -\frac{g}{4\Omega^2}\sin 2\Theta, \quad y' = -\frac{g}{4\Omega^2}(1 - \cos 2\Theta), \quad z' = -\frac{2g\tan a}{\Omega^2}(1 - \cos\Theta), \quad . \quad (170)$$

since $dt = d\Theta/\Omega$.

Take a circular cylinder of diameter $g/2\Omega^2$, touching the xz plane along the Oz axis, Fig. 67 (a). Let this be carried down the average straight-line path without rotation. Then the centre of gravity runs

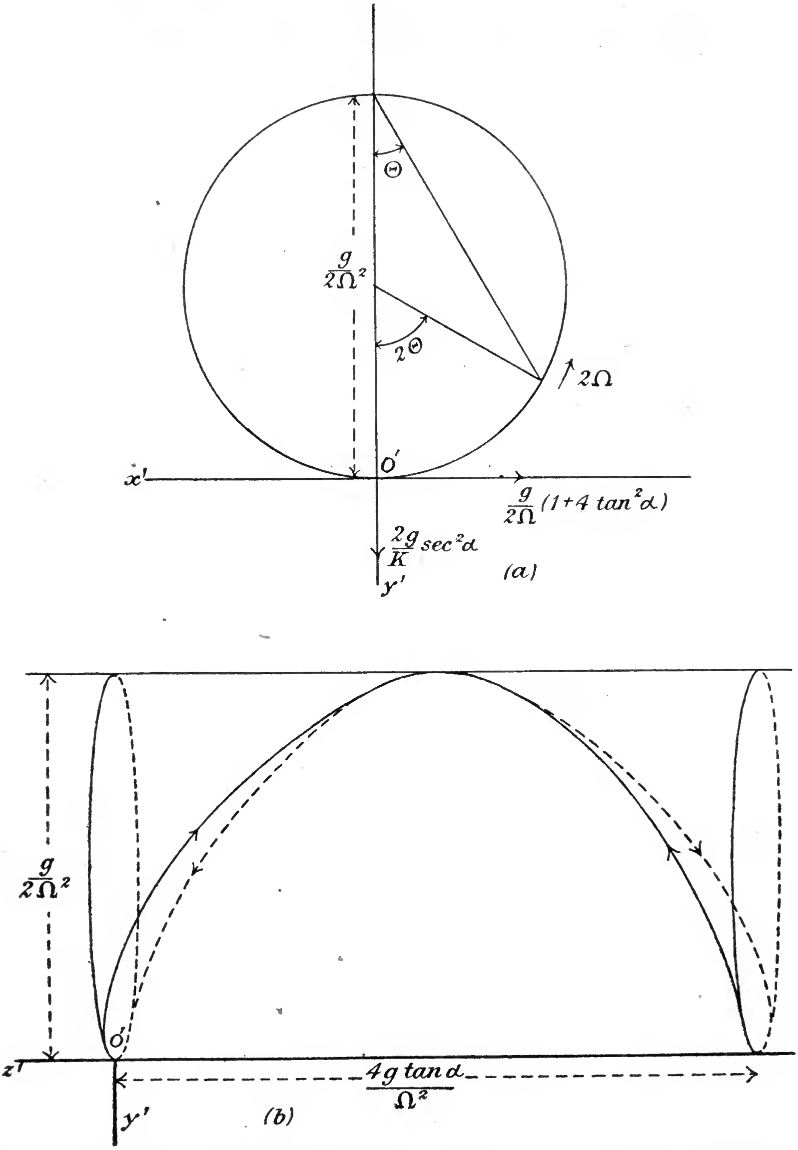

FIG. 67.—Lamina with Simplified Law of Air-Resistance. General Case with Axis of Rotation Horizontal. Motion of Centre of Gravity.

round this cylinder in a closed double loop (Fig. 67 (b)) obtained by the intersection of this circular cylinder with another parabolic cylinder of equation

$$y' + \frac{g}{2\Omega^2} = \frac{\Omega^2}{8g\tan^2 a}\left(z' + \frac{2g\tan a}{\Omega^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

This relative path depends only on Ω, and not on K; the centre of gravity runs round the axis of the cylinder with an angular velocity 2Ω.

When $\tan a = 0$ we go back to the equations (170) and we get $z' = 0$, this being, in fact, the problem of § 130. The relative path of the centre of gravity referred to the hypothetical uniformly moving point is now a circle described uniformly. The curves in Figs. 43–44 are, in fact, the curves obtained when a circle is made to move, without rotation, along a straight line, whilst a point is made to move round the circle with constant speed.

136. **General Direction of the Axis of Rotation.**—It will be found that in the general case the motion of the centre of gravity, relatively to a point coincident with it initially ($\Theta = 0$, $t = 0$) and moving along the average straight-line path, can be represented by means of a curve given by the intersection of a parabolic cylinder with a cone of the second degree.

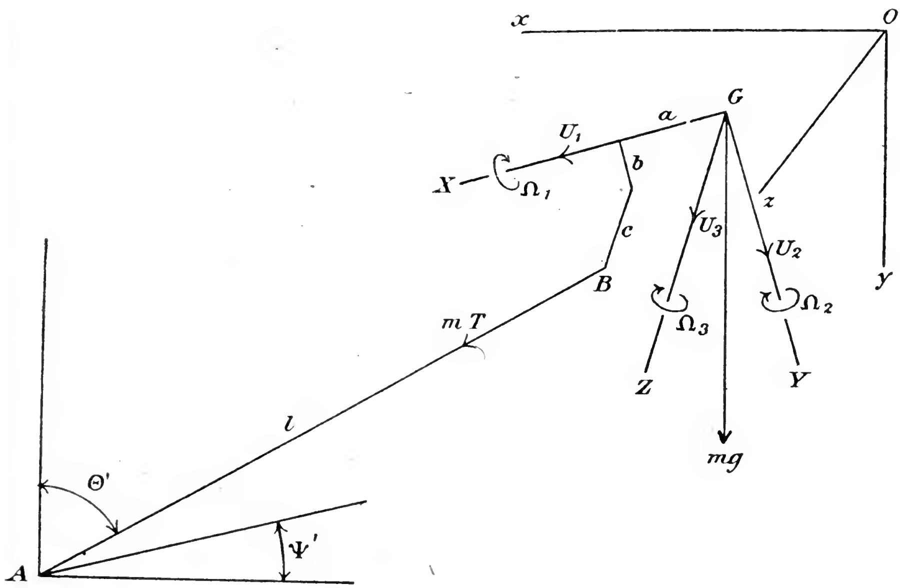

FIG. 68.—The Kite: Three-Dimensional Problem

137. **The Kite; Three-Dimensional Problem.**—The longitudinal kite problem of §§ 93–97 can be extended into three dimensions by introducing the necessary modifications in the notation and in the equations of motion. Let the body axes in Fig. 68 be derived from axes in space by means of displacements Ψ, Θ, Φ, as in Figs. 55–57. Let B, the point of attachment of the string (assumed single) to the kite, have co-ordinates (a, b, c); let the string have length l, and let its direction in space be indicated by means of the angles Θ' (zenith distance) and Ψ' (azimuth from some standard vertical plane). Θ' is the angle between AB and the upward vertical; Ψ' is the angle between the vertical plane containing the string and the vertical plane Oxy. The mass of the kite is m.

If mT is the tension of the string, then its components along the space axes are

$$mT \sin \Theta' \cos \Psi', \quad mT \cos \Theta', \quad mT \sin \Theta' \sin \Psi' \quad . \quad . \quad . \quad . \quad (172)$$

To obtain the components along, and the moments about, the body axes, we use Figs. 55–57 as when we argued out the gravity components. We find that the components along the body axes are obtained by multiplying the quantities (172) by the following sets of direction cosines, and adding :—

For the X component use

$$\cos \Theta \cos \Psi, \quad \sin \Theta, \quad -\cos \Theta \sin \Psi;$$

for the Y component use

$$-\sin \Theta \cos \Phi \cos \Psi + \sin \Phi \sin \Psi, \quad \cos \Theta \cos \Phi, \quad \sin \Theta \cos \Phi \sin \Psi + \sin \Phi \cos \Psi;$$

for the Z component use

$$\sin \Theta \sin \Phi \cos \Psi + \cos \Phi \sin \Psi, \quad -\cos \Theta \sin \Phi, \quad -\sin \Theta \sin \Phi \sin \Psi + \cos \Phi \cos \Psi. \quad (173)$$

The moments, due to the tension, about the body axes are :

$$\left.\begin{aligned} X \text{ couple} &= b\,(Z \text{ component}) - c\,(Y \text{ component}), \\ Y \text{ couple} &= c\,(X \text{ component}) - a\,(Z \text{ component}), \\ Z \text{ couple} &= a\,(Y \text{ component}) - b\,(X \text{ component}). \end{aligned}\right\} \quad . \quad . \quad . \quad (174)$$

138. **Stability.**—That equilibrium is possible in general is shown as in § 94. Suppose there is a small disturbance, and assume that we can neglect squares, products, etc., of the small quantities representing this disturbance and its effects. Let Θ_0, Φ_0, Ψ_0, Θ_0', Ψ_0', T_0 refer to the equilibrium position. In the motion caused by the disturbance, let

$$U_1 = u_1, \quad U_2 = u_2, \quad U_3 = u_3, \quad \Omega_1 = \omega_1, \quad \Omega_2 = \omega_2, \quad \Omega_3 = \omega_3, \quad \Theta = \Theta_0 + \theta, \quad \Phi = \Phi_0 + \phi,$$
$$\Psi = \Psi_0 + \psi, \quad \Theta' = \Theta'_0 + \theta', \quad \Psi' = \Psi'_0 + \psi', \quad T = T_0 + \delta T.$$

We take here the case of a symmetrical kite undergoing a general disturbance, the plane XZ being the plane of symmetry in the kite, and this plane being vertical in the equilibrium position. Hence $\Phi_0 = 0$. We can, without loss of generality, make $\Psi_0 = \Psi_0' = 0$, *i.e.* we measure the azimuth from the equilibrium position of the plane of symmetry. The components (172) become, to the first order of small quantities,

$$mT \sin \Theta', \quad mT \cos \Theta', \quad mT_0 \sin \Theta'_0 \,.\, \psi'.$$

The factors (173) become

$$\begin{gathered} \cos \Theta, \quad \sin \Theta, \quad -\cos \Theta_0 \,.\, \psi, \\ -\sin \Theta, \quad \cos \Theta, \quad \sin \Theta_0 \,.\, \psi + \phi, \\ \sin \Theta_0 \,.\, \phi + \psi, \quad -\cos \Theta_0 \,.\, \phi, \quad 1. \end{gathered}$$

We, therefore, find for the X, Y, Z components of the tension the following values to the first order of small quantities :

$$\left.\begin{aligned} & mT \sin (\Theta' + \Theta), \\ & mT \cos (\Theta' + \Theta), \\ & -mT_0 \cos (\Theta'_0 + \Theta_0)\,\phi + mT_0 \sin \Theta'_0 (\psi + \psi'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (175)$$

We see that the X, Y components *do not involve* ϕ, ψ, *and* ψ' *at all*, whilst the Z couple (see equations (174)) also possesses this property. On the other hand, since in the case of symmetry $c = 0$, the Z component and the X, Y couples involve only the variables ϕ, ψ, and ψ', *not* θ, θ'.

Again, consider the air pressures. As in § 95, we take the wind to have component velocities U_1', U_2' parallel to the negative directions of the body axes in the position of equilibrium. ($U_3' = 0$, because of the assumed symmetry.) If the body axes have turned through small angles ϕ_1, θ_1, ψ_1, where

$$\omega_1 = \frac{d\phi_1}{dt}, \quad \omega_2 = \frac{d\psi_1}{dt}, \quad \omega_3 = \frac{d\theta_1}{dt}, \qquad \qquad (176)$$

then the new component wind velocities are (similarly to § 103)

$$U_1' - \theta_1 U_2', \quad U_2' + \theta_1 U_1', \quad \phi_1 U_2' - \psi_1 U_1'.$$

(Our θ_1 is what we called θ in the two-dimensional work, § 95). It should be noted that ϕ_1, θ_1, ψ_1 are not quite the same as ϕ, θ, ψ. The latter are given in terms of ω_1, ω_2, ω_3 by means of the equations (137), which now become

$$\omega_1 = \frac{d\phi}{dt} + \sin\Theta_0 \frac{d\psi}{dt}, \quad \omega_2 = \frac{d\psi}{dt}, \quad \omega_3 = \frac{d\theta}{dt}, \qquad \qquad (177)$$

since $\Theta = \Theta_0$, $\Phi_0 = 0$ in the equilibrium position.

The air forces and couples are given by expressions like

$$\begin{aligned} R_1 &\equiv R_1(U_{10} + u_1 - \theta_1 U_{20},\ U_{20} + u_2 + \theta_1 U_{10},\ u_3 + \phi_1 U_{20} - \psi_1 U_{10},\ \omega_1,\ \omega_2,\ \omega_3) \\ &\equiv R_{10}(U_{10},\ U_{20},\ 0,\ 0,\ 0,\ 0) \\ &\qquad + U(a_x \,.\, \overline{u_1 - \theta_1 U_{20}} + b_x \,.\, \overline{u_2 + \theta_1 U_{10}} + c_x \,.\, \overline{u_3 + \phi_1 U_{20} - \psi_1 U_{10}} \\ &\qquad\qquad + d_x\omega_1 + e_x\omega_2 + f_x\omega_3), \end{aligned}$$

where R_{10} is the same as in the longitudinal problem, and a_x, b_x, etc., are the resistance derivatives exactly as in the aeroplane investigations. Similar results hold for R_2, R_3, G_1, G_2, G_3; U is once more $(U_{10}^2 + U_{20}^2)^{\frac{1}{2}}$.

Further, by symmetry we can say *a priori* that the resistance derivatives of R_1, R_2, G_3 with respect to u_3, ω_1, ω_2, and of R_3, G_1, G_2 with respect to u_1, u_2, ω_3, must vanish. Hence we have

$$\begin{aligned} R_1 &= R_{10} + U(a_x \,.\, \overline{u_1 - \theta_1 U_{20}} + b_x \,.\, \overline{u_2 + \theta_1 U_{10}} + f_x\omega_3), \\ R_2 &= R_{20} + U(a_y \,.\, \overline{u_1 - \theta_1 U_{20}} + b_y \,.\, \overline{u_2 + \theta_1 U_{10}} + f_y\omega_3), \\ R_3 &= R_{30} + U(c_z \,.\, \overline{u_3 + \phi_1 U_{20} - \psi_1 U_{10}} + d_z\omega_1 + e_z\omega_2), \\ G_1 &= G_{10} + U(c_1 \,.\, \overline{u_3 + \phi_1 U_{20} - \psi_1 U_{10}} + d_1\omega_1 + e_1\omega_2), \\ G_2 &= G_{20} + U(c_2 \,.\, \overline{u_3 + \phi_1 U_{20} - \psi_1 U_{10}} + d_2\omega_1 + e_2\omega_2), \\ G_3 &= G_{30} + U(a_3 \,.\, \overline{u_1 - \theta_1 U_{20}} + b_3 \,.\, \overline{u_2 + \theta_1 U_{10}} + f_3\omega_3). \end{aligned}$$

Finally, the gravity terms per unit mass are

$$g(\sin\Theta_0 + \theta\cos\Theta_0), \quad g(\cos\Theta_0 - \theta\sin\Theta_0), \quad -g\cos\Theta_0 \,.\, \phi.$$

If now we write down the equations of motion (136), in which the propeller thrust is replaced by the components and couples due to the tension, we get (using the obvious conditions of equilibrium, which are left to the student)

$$\left.\begin{aligned}
\frac{du_1}{dt} &= g\cos\Theta_0\,.\,\theta + T_0\cos(\Theta_0'+\Theta_0)(\theta'+\theta) + \delta T\,.\sin(\Theta_0'+\Theta_0) \\
&\qquad - U(a_x\,.\,\overline{u_1-\theta_1 U_{20}} + b_x\,.\,\overline{u_2+\theta_1 U_{10}} + f_x\omega_3), \\
\frac{du_2}{dt} &= -g\sin\Theta_0\,.\,\theta - T_0\sin(\Theta_0'+\Theta_0)(\theta'+\theta) + \delta T\,.\cos(\Theta_0'+\Theta_0) \\
&\qquad - U(a_y\,.\,\overline{u_1-\theta_1 U_{20}} + b_y\,.\,\overline{u_2+\theta_1 U_{10}} + f_y\omega_3), \\
\frac{du_3}{dt} &= -g\cos\Theta_0\,.\,\phi - T_0\cos(\Theta_0'+\Theta_0)\,.\,\phi + T_0\sin\Theta_0'\,.\,(\psi+\psi') \\
&\qquad - U(c_z\,.\,\overline{u_3+\phi_1 U_{20}-\psi_1 U_{10}} + d_z\omega_1 + e_z\omega_2), \\
\frac{d\omega_1}{dt} - \frac{F}{A}\frac{d\omega_2}{dt} &= -\frac{bT_0}{k_1^2}\cos(\Theta_0'+\Theta_0)\,.\,\phi + \frac{bT_0}{k_1^2}\sin\Theta_0'\,.\,(\psi+\psi') \\
&\qquad - U(c_1\,.\,\overline{u_3+\phi_1 U_{20}-\psi_1 U_{10}} + d_1\omega_1 + e_1\omega_2), \\
\frac{d\omega_2}{dt} - \frac{F}{B}\frac{d\omega_1}{dt} &= \frac{aT_0}{k_1^2}\cos(\Theta_0'+\Theta_0)\,.\,\phi - \frac{aT_0}{k_1^2}\sin\Theta_0'\,.\,(\psi+\psi') \\
&\qquad - U(c_2\,.\,\overline{u_3+\phi_1 U_{20}-\psi_1 U_{10}} + d_2\omega_1 + e_2\omega_2), \\
\frac{d\omega_3}{dt} &= \frac{a}{k_3^2}\{-T_0\sin(\Theta_0'+\Theta_0)(\theta+\theta') + \delta T\,.\cos(\Theta_0'+\Theta_0)\} \\
&\qquad - \frac{b}{k_3^2}\{T_0\cos(\Theta_0'+\Theta_0)(\theta'+\theta) + \delta T\,.\sin(\Theta_0'+\Theta_0)\} \\
&\qquad - U(a_3\,.\,\overline{u_1-\theta_1 U_{20}} + b_3\,.\,\overline{u_2+\theta_1 U_{10}} + f_3\omega_3),
\end{aligned}\right\}\quad (178)$$

where $A = mk_1^2$, $B = mk_2^2$, $C = mk_3^2$ are the three moments of inertia about the body axes, and F is the XY product of inertia; the products D, E are zero by symmetry.

We have, in addition, some geometrical conditions, given by the fact that the kite is attached to a point by an inextensible string of given length l. The velocities of the point of attachment B, referred to the body axes, are (see § 104)

$$u_1 - \omega_3 b,\quad u_2 + \omega_3 a,\quad u_3 + \omega_1 b - \omega_2 a,$$

since $c = 0$ in our symmetrical problem. These being small quantities, we can ignore the small changes in the directions to which they are referred. But the velocity of B must be compounded of $l\dfrac{d\theta'}{dt}$ perpendicular to the string, downwards, *i.e.* Θ' increasing, and $l\sin\Theta'_0\,.\,\dfrac{d\psi'}{dt}$ horizontally and perpendicular to the plane of symmetry, along Ψ' increasing. Hence we have

$$\left.\begin{aligned}
(u_1 - b\omega_3)\sin(\Theta_0'+\Theta_0) + (u_2 + a\omega_3)\cos(\Theta_0'+\Theta_0) &= 0, \\
-(u_1 - b\omega_3)\cos(\Theta_0'+\Theta_0) + (u_2 + a\omega_3)\sin(\Theta_0'+\Theta_0) &= l\frac{d\theta'}{dt} = l\frac{d}{dt}(\theta'+\theta) - l\frac{d\theta}{dt}, \\
u_3 + \omega_1 b - \omega_2 a &= -l\sin\Theta_0'\,.\,\frac{d\psi'}{dt}.
\end{aligned}\right\}\quad (179)$$

We now have nine equations (178) and (179) for the unknown quantities

$$u_1,\ u_2,\ u_3,\ \theta,\ \phi,\ \psi,\ \theta',\ \psi',\ \delta T.$$

There are no other independent unknowns, since ω_1, ω_2, ω_3 are given by θ, ϕ, ψ, and θ_1, ϕ_1, ψ_1 are expressible in terms of θ, ϕ, ψ by means of (176) and (177). We have, therefore, given a complete analytical statement of our problem.

139. It is now clear that the general problem of the stability of the kite *can be split up into two independent problems*, when there is symmetry in the equilibrium position. In equations (178) the first, second, and sixth equations involve only u_1, u_2, ω_3, θ, θ'; whilst the third, fourth, and fifth involve only u_3, ω_1, ω_2, ϕ, ψ, ψ'. Also in (179) the first two involve only u_1, u_2, ω_3, θ, θ', whilst the third only involves u_3, ω_1, ω_2, ψ'. Further, from (176–77) we have

$$\frac{d\theta_1}{dt}=\frac{d\theta}{dt},\quad \frac{d\phi_1}{dt}=\frac{d\phi}{dt}+\sin\Theta_0\,.\,\frac{d\psi}{dt},\quad \frac{d\psi_1}{dt}=\frac{d\psi}{dt},$$

so that to the first order of small quantities we may write

$$\theta=\theta_1,\quad \phi=\phi_1-\sin\Theta_0\,.\,\psi_1,\quad \psi=\psi_1.$$

Consequently the longitudinal stability is conditioned by exactly the same equations as in the two-dimensional work, and we can say that the longitudinal stability of the symmetrical kite is, in the general displacement, dependent on the conditions given by the analysis of § 95.

The lateral stability is determined by the following equations, in which $a\equiv\Theta_0'+\Theta_0$:

$$\begin{aligned}\frac{du_3}{dt}=&-g\cos\Theta_0\,.\,(\phi_1-\sin\Theta_0\,.\,\psi_1)-T_0\cos a\,.\,(\phi_1-\sin\Theta_0\,.\,\psi_1)+T_0\sin\Theta_0'\,.\,(\psi_1+\psi')\\ &\qquad-U\left(c_z\,.\,\overline{u_3+\phi_1U_{20}-\psi_1U_{10}}+d_z\frac{d\phi_1}{dt}+e_z\frac{d\psi_1}{dt}\right),\\ \frac{d^2\phi_1}{dt^2}-\frac{F}{A}\frac{d^2\psi_1}{dt^2}=&-\frac{bT_0}{k_1^2}\cos a\,.\,(\phi_1-\sin\Theta_0\,.\,\psi_1)+\frac{bT_0}{k_1^2}\sin\Theta_0'\,.\,(\psi_1+\psi')\\ &\qquad-U\left(c_1\,.\,\overline{u_3+\phi_1U_{20}-\psi_1U_{10}}+d_1\frac{d\phi_1}{dt}+e_1\frac{d\psi_1}{dt}\right),\\ \frac{d^2\psi_1}{dt^2}-\frac{F}{B}\frac{d^2\phi_1}{dt^2}=&\ \frac{aT_0}{k_2^2}\cos a\,.\,(\phi_1-\sin\Theta_0\,.\,\psi_1)-\frac{aT_0}{k_2^2}\sin\Theta_0'\,.\,(\psi_1+\psi')\\ &\qquad-U\left(c_2\,.\,\overline{u_3+\phi_1U_{20}-\psi_1U_{10}}+d_2\frac{d\phi_1}{dt}+e_2\frac{d\psi_1}{dt}\right),\end{aligned}$$

$$u_3+b\frac{d\phi_1}{dt}-a\frac{d\psi_1}{dt}=-l\sin\Theta_0'\,.\,\frac{d\psi'}{dt}.$$

The device adopted in § 94, namely, the choice of the direction GX parallel to the direction of the string, makes $a=90°$, so that $\sin a=1$, $\cos a=0$; also $\sin\Theta_0'=\cos\Theta_0$. Using the method of § 85, we deduce that the kite is stable laterally if the solutions of the following equation in λ have their real parts negative:

$$\begin{vmatrix}\lambda+c_z, & d_z\lambda+\frac{U_{20}}{U}c_z+\frac{g}{U^2}\cos\Theta_0, & e_z\lambda-\frac{U_{10}}{U}c_z-\frac{g}{U^2}\sin\Theta_0\cos\Theta_0, & 1\\ c_1, & \lambda^2+d_1\lambda+\frac{U_{20}}{U}c_1, & -\frac{F}{A}\lambda^2+e_1\lambda-\frac{U_{10}}{U}c_1, & \frac{b}{k_1^2}\\ c_2, & -\frac{F}{B}\lambda^2+d_2\lambda+\frac{U_{20}}{U}c_2, & \lambda^2+e_2\lambda-\frac{U_{10}}{U}c_2, & -\frac{a}{k_2^2}\\ 1, & b\lambda & -(a+l\sin\Theta'_0)\lambda, & -\frac{U^2l}{T_0}\lambda\end{vmatrix}=0.$$

140 **Summary of Assumptions Made in the Aeroplane Theory.**—It is useful to bear in mind the things we have neglected in all our work. As already remarked in § 38, we have assumed the air forces to be obtained on the basis of steady motion at any instant. At present there is little hope that we shall be able to get rid of this simplification. We have assumed the aeroplane to be a rigid body of constant weight. Actually it is not quite rigid, and it contains relatively movable parts; the propeller and the mechanism of the engine, the passengers, the fuel, etc. We have assumed the density of the air constant. For stability work this is of no importance. In the general problem the assumption implies limitation on any solution obtained, which is only true for a comparatively thin layer of air having the density adopted in the calculations. In long climbing and descending flights the density of the air plays an important part. The density is reduced to one-half at a height of three or four miles, and modern flight is even beyond this. It follows that the general steady motions we have discussed are not correct: as the aeroplane rises or sinks, the appropriate steady motion varies. Finally, the fuel diminishes in weight, and this means a continual lightening of the machine.

In our treatment we have also assumed the propeller thrust to pass through the centre of gravity. This is nearly, but not quite, true in all machines. If the centre of gravity is above, or below, the thrust, we get a couple due to the latter, and this must be taken account of in all the work. The conditions for steady motion now involve the condition that the moment of the thrust is just balanced by the moment of the air forces. This is expressed by stating that the resultant air pressure must pass through the point of intersection of the propeller axis, and the vertical through the centre of gravity. We get similar arguments to those in §§ 66–70, etc., but more complicated.

There are machines with more than one propeller. When there are two propellers they are as nearly equal as possible and placed symmetrically in the machine. In longitudinal steady motion no complication is hereby introduced. In the general case, however, the propellers will not necessarily give equal thrusts, and this must be remembered in the corresponding investigations.

Finally, a word about the axes chosen. We have preferred in this book to make the X axis always coincide (or be parallel to) the propeller axis. In many investigations hitherto published the X axis has been made to coincide in direction with the (backward) direction of flight. We have here adopted the former choice for two reasons. The first is that we thus fix the body axes once for all, and we understand more clearly the definitions of the various quantities involved, particularly the resistance derivatives in steady motions of various types. Secondly, in non-symmetrical flight the X axis would be out of the generally defined longitudinal plane, if it were made to depend on the direction of steady flight. It will, nevertheless, be a good exercise for the student to develop the theory of longitudinal stability with the X axis along the direction of steady flight. In the practical aerodynamical work there is justification for this position of the X axis, because of the types of experiments that have to be made (see the *Reports of Advisory Committee for Aeronautics*).

APPENDIX TO CHAPTER IV

In the researches published by the workers at the National Physical Laboratory, and others, the axes used in three-dimensional aeroplane dynamics, corresponding to those given in the Appendix to Chapter III., are indicated in Fig. 49A, which should be compared with Fig. 49. Referring to Fig. 33A, we see that now an additional axis Oy is put in, forming a left-handed system with Ox, Oz; similarly an additional axis

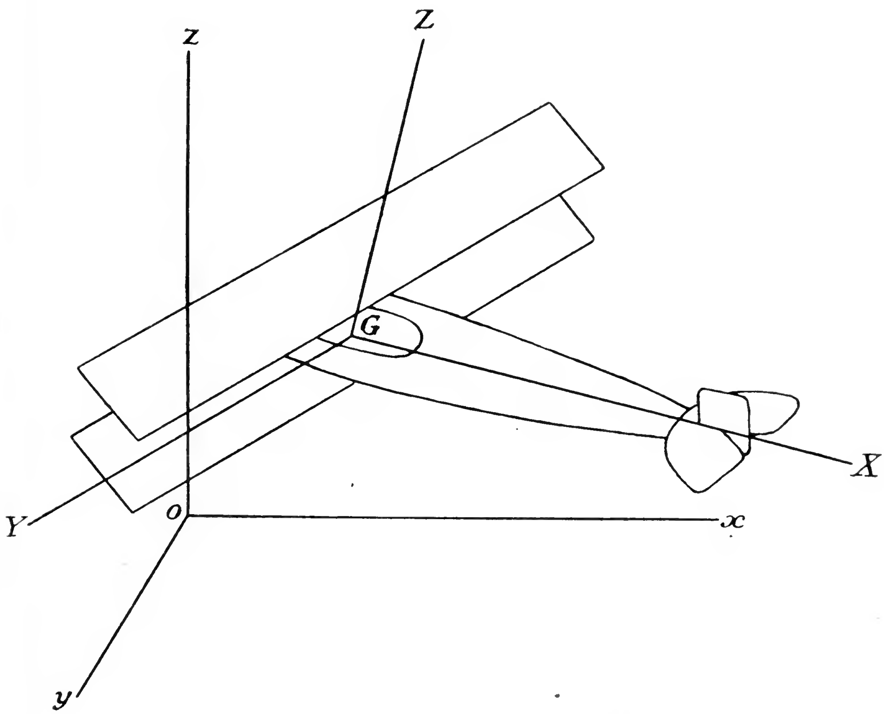

FIG. 49A.—Space Axes and Body Axes: Alternative System.

GY is put in, so that GX, Y, Z form a left-handed system. As seen by the pilot, GX is directly backwards, GY is to the left, and GZ is vertically upwards.

The notation is indicated in Fig. 8A. The velocity components of the aeroplane along the body axes GX, Y, Z are U, V, W respectively, whilst the angular velocity components about these axes are P, Q, R, the senses being as indicated in Fig. 8A, *i.e.*, U and P form a left-handed screw in the standard case, and similarly for V and Q, W and R. The forces due to the air-pressures are components X, Y, Z per unit mass of

the aeroplane, along the directions of GX, GY, GZ respectively, in the same senses as U, V, W; the couples are L, M, N per unit moment of inertia about each axis GX, GY, GZ respectively, in the same senses as P, Q, R. Then if m is the mass of the aeroplane and A, B, C are the moments of inertia about the axes GX, GY, GZ, then the components of air-force are mX, mY, mZ, and the components of air-couple are AL, BM, CN. As before, the products of inertia are D, E, F.

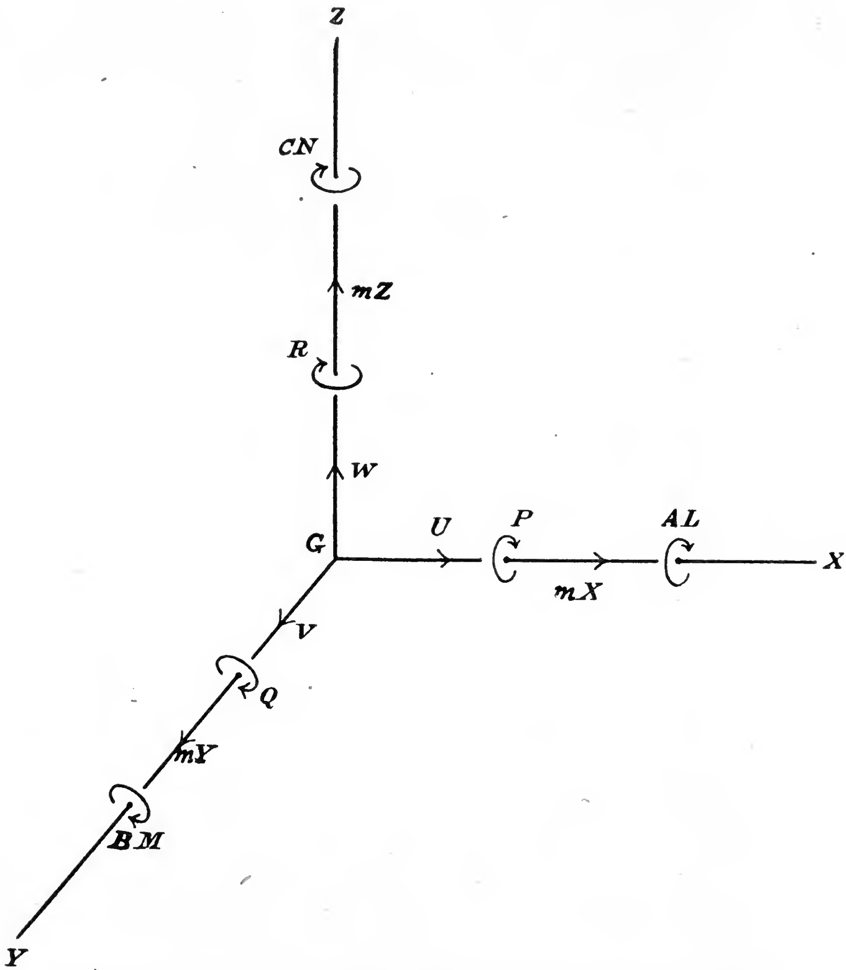

FIG. 8A.—Body Axes, Velocities and Rotations; Air Forces and Couples: Alternative Notation.

In order to pass from the space axes Ox, y, z to the body axes GX, Y, Z we make use of the angles Θ, Φ, Ψ, defined in the following manner.

Let GX, Y, Z coincide initially (in direction) with Ox, y, z. Let there be a **yaw** through an angle Ψ about the vertical Z axis, measured positive from X to Y. Then let there be a **pitch** through an angle Θ about the *new* Y axis, measured positive from Z to X. Finally let there

be a **roll** through an angle Φ about the X axis *thus obtained*, measured positive from Y to Z.

The components of gravity along the X, Y, Z axes are, per unit mass,

$$g \sin \Theta, \quad -g \cos \Theta \sin \Phi, \quad -g \cos \Theta \cos \Phi.$$

The relations between P, Q, R and the variations in Θ, Φ, Ψ are:

$$\begin{aligned} P &= \sin \Psi \frac{d\Theta}{dt} + \cos \Theta \cos \Psi \frac{d\Phi}{dt}, \\ Q &= \cos \Psi \frac{d\Theta}{dt} - \cos \Theta \sin \Psi \frac{d\Phi}{dt}, \\ R &= \sin \Theta \frac{d\Phi}{dt} + \frac{d\Psi}{dt}. \end{aligned}$$

It is left as an exercise to the student to write down the equations of motion similar to (136) in § 107.

When there is a small disturbance from a steady motion, we write

$$U = U_0 + u, \quad V = V_0 + v, \quad W = W_0 + w, \quad P = P_0 + p, \quad Q = Q_0 + q, \quad R = R_0 + r,$$
$$\Theta = \Theta_0 + \theta, \quad \Phi = \Phi_0 + \phi, \quad \Psi = \Psi_0 + \psi;$$

the resistance derivatives are then defined by the symbolism:

$$\begin{aligned} X &= X_0 + uX_u + vX_v + wX_w + pX_p + qX_q + rX_r, \\ Y &= Y_0 + uY_u + vY_v + wY_w + pY_p + qY_q + rY_r, \\ Z &= Z_0 + uZ_u + vZ_v + wZ_w + pZ_p + qZ_q + rZ_r, \\ L &= L_0 + uL_u + vL_v + wL_w + pL_p + qL_q + rL_r, \\ M &= M_0 + uM_u + vM_v + wM_w + pM_p + qM_q + rM_r, \\ N &= N_0 + uN_u + vN_v + wN_w + pN_p + qN_q + rN_r, \end{aligned}$$

it being assumed that the propeller thrust and its derivatives are absorbed in X, Y, Z, L, M, N.

Taking the case of the symmetrical aeroplane, whose motion is slightly divergent from a rectilinear steady motion, § 114, it is at once seen that the equations of motion split up into two mutually independent sets: one set of three equations is identical with the equations given in the Appendix to Chapter III. The other set of three equations gives the assumed small *lateral oscillations*, and are as follows:—

$$\begin{aligned} &\frac{dv}{dt} - W_0 \frac{d\phi}{dt} + W_0 \sin \Theta_0 \frac{d\psi}{dt} + U_0 \cos \Theta_0 \frac{d\psi}{dt} \\ &\qquad = -g\phi \cos \Theta_0 + vY_v + \left(\frac{d\phi}{dt} - \sin \Theta_0 \frac{d\psi}{dt}\right) Y_p + \cos \Theta_0 \frac{d\psi}{dt} Y_r, \\ &\frac{d^2\phi}{dt^2} - \sin \Theta_0 \frac{d^2\psi}{dt^2} - \frac{E}{A} \cos \Theta_0 \frac{d^2\psi}{dt^2} \\ &\qquad = vL_v + \left(\frac{d\phi}{dt} - \sin \Theta_0 \frac{d\psi}{dt}\right) L_p + \cos \Theta_0 \frac{d\psi}{dt} L_r, \\ &\cos \Theta_0 \frac{d^2\psi}{dt^2} - \frac{E}{C}\left(\frac{d^2\phi}{dt^2} - \sin \Theta_0 \frac{d^2\psi}{dt^2}\right) \\ &\qquad = vN_v + \left(\frac{d\phi}{dt} - \sin \Theta_0 \frac{d\psi}{dt}\right) N_p + \cos \Theta_0 \frac{d\psi}{dt} N_r. \end{aligned}$$

We deduce for the lateral motion the biquadratic determinantal equation

$$\begin{vmatrix} \lambda - Y_v, & -(W_0 + Y_p)\lambda + g\cos\Theta_0, & (W_0 + Y_p)\sin\Theta_0 + (U_0 - Y_r)\cos\Theta_0, \\ -L_v, & \lambda^2 - L_p\lambda, & -(\sin\Theta_0 + \frac{E}{A}\cos\Theta_0)\lambda + L_p\sin\Theta_0 - L_r\cos\Theta_0, \\ -N_v, & -\frac{E}{C}\lambda^2 - N_p\lambda, & (\cos\Theta_0 + \frac{E}{C}\sin\Theta_0)\lambda + N_p\sin\Theta_0 - N_r\cos\Theta_0 \end{vmatrix} = 0.$$

This can be simplified, as in § 117, and becomes

$$\begin{vmatrix} \lambda - Y_v, & -(W_0 + Y_p)\lambda + g\cos\Theta_0, & (U_0 - Y_r)\lambda + g\sin\Theta_0 \\ -L_v, & \lambda^2 - L_p\lambda, & -\frac{E}{A}\lambda^2 - L_r\lambda \\ -N_v, & -\frac{E}{C}\lambda^2 - N_p\lambda, & \lambda^2 - N_r\lambda \end{vmatrix} = 0.$$

If this determinant is developed, and the spurious factor λ taken out, we get an equation of the form

$$A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2 = 0,$$

and the conditions of stability are that

$$A_2, B_2, C_2, D_2, E_2, \text{ and } H_2 \equiv B_2C_2D_2 - A_2D_2^2 - E_2B_2^2$$

all have the same sign.

In the notation adopted by the Technical Terms Committee of the Royal Aeronautical Society, the spaces axes Ox, y, z and the body axes GX, Y, Z are taken in opposite directions to those of the axes here defined.

EXERCISES (Chapter IV)

1. Write out the equations of motion for motion slightly divergent from general rectilinear steady motion, with the aeroplane not necessarily symmetrical, and deduce the octic determinantal equation which governs the stability.

2. Generalise the work in Question 1 to the most general steady motion when there is a small disturbance.

3. Investigate the effect of the gyroscopic couple due to the propeller, and the engine if of rotary type: assume that the derivatives are obtained by considering variations of the couples when the motion is steady.

4. Investigate the necessary and sufficient conditions that the real parts of the roots of the equation

$$a_0 + a_1\lambda + a_2\lambda^2 + \quad + a_n\lambda^n = 0$$

shall all be negative: take all the coefficients real.

(*Note.*—Put $\lambda = \iota t$, then in the t plane of an Argand diagram the roots are all to be represented by points lying above the real axis. The equation is now

$$x + \iota y \equiv a_0 - a_2t^2 + a_4t^4 - \quad + \iota(a_1t - a_3t^3 + \ldots) = 0.$$

It is at once seen that amp $(x + \iota y)$ must increase by $n\pi$ as t assumes all real values from $+\infty$ to $-\infty$.)

5. Find the path of the centre of gravity of the lamina discussed in §§ 127–136, taking the general case as suggested in § 136.

6. Assuming that the air-resistance for any element of area of a lamina is normal and varies as the normal velocity component, investigate the general motion of such a lamina

(i) under no forces (except the air-resistance);

(ii) under gravity.

7. Investigate the stability of a parachute on the assumptions of Question 6.

8. Examine particularly the motion in Question 6 when the lamina is a circle or square or of any other shape possessing such symmetry as regards moments of inertia, the centre of gravity being at the centre of figure.

SECTION II

DYNAMICS OF AIR

CHAPTER V

AERODYNAMICS: PERFECT FLUID

141. **Restriction to Non-Viscous Fluid.**—The student will readily appreciate the necessity for a strict limitation of the generality of the resistance problem when attacked by mathematical analysis. Regarding the subject *a priori*, we at once decide that any attempt to include the effect of viscosity must be postponed to a later stage. We must first examine how the problem of air resistance shapes itself on the simplest possible assumptions, and it is clear that the omission of viscosity must conduce to simplification. The question at once arises: Is there any justification for this abstraction? All fluids are more or less viscous, and surely a solution based on the absence of viscosity can be of very limited value. The best answer to this objection is afforded by a parallel case in dynamics. In elementary dynamics it is an advantage to omit friction and resistance in general, and during a considerable portion of a course in mechanics it is usual to develop the subject on such a simplified hypothesis: the experience gained by this simplified solution then helps us in the attack of the more ambitious problems. Similarly, in our present subject it is inevitable that for some considerable time only very simplified hypotheses will lead to soluble problems: we must wait till such experience has supplied us with better analytical methods before we can attack the more practical cases.

Restriction to Two-Dimensional Steady Motion.—It will save us much trouble and unnecessary, because useless, symbolical verbiage if we at once state that one of the restrictions that we introduce into mathematical aerodynamics is the limitation of the motion to two dimensions. The restriction is not very arbitrary when considered from the point of view of the aeroplane wings.

Aeroplane wings are made, more or less, according to the plan indicated in Fig. 69 (*a*). For most of the span the section is practically constant, and it cannot be very far from the truth to assume that the motion of the air between a pair of planes perpendicular to the wings is the same for all positions along the span. We assume, in fact, that for the greater part of the span the motion of the air in any plane perpendicular to the wings is the same. This means in addition that any particle of air moves in a plane perpendicular to the wings, so that all the air particles in any such plane remain in this plane all through the motion. But we are not dealing with the individual molecules of the air: rather with "drops" of air. We shall then consider the motion of the air between two parallel

planes (Fig. 69 (*b*)), which we can consider vertical and at unit distance apart, it being assumed that the air moves as if it consisted of an infinite number of rods of unit length, each of which remains straight and perpendicular to the planes, *i.e.* in the case of symmetrical or longitudinal flight these rods remain horizontal.

We further suppose the motion to be steady, as already explained in Section I, § 38. If at any instant we take a sort of mental photograph

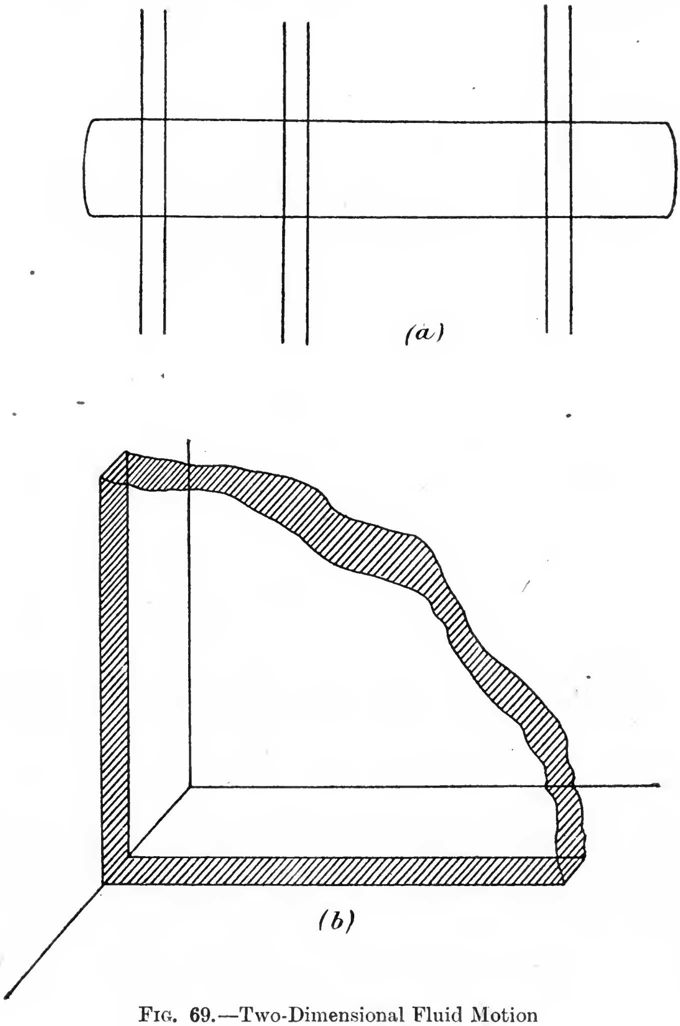

Fig. 69.—Two-Dimensional Fluid Motion

of the state of motion of the air, and then take another such photograph some time later, the two are supposed to be indistinguishable, *i.e.* in each the motion of the fluid at any point is the same, and the concentration of the matter is the same, although really it is not the same matter in the two photographs. If the fluid is homogeneous, the steadiness of the motion is characterised merely by the fact that at any given point the matter moves in a fixed manner, so that the motion is the same in both photographs. If

now we trace the path of any given small drop of the fluid, the tangent to this path at any point represents the direction of the steady motion of the fluid at this point at any time. We can draw a number of such curves, filling the whole of the space occupied by the fluid: they are called the **Stream Lines.** We shall return to these later.

But it is necessary here to explain more clearly the meaning of the assumption of steady motion. The student will easily see that if we take the ordinary problem of an aeroplane moving through air there is really no steady motion of the air possible. For a point which is fixed in space is not fixed relatively to the aeroplane, so that at any given point we do not get the same motion at all times. We therefore extend the idea of steady motion to refer also to motion of fluid relatively to a uniformly moving rigid boundary. Assuming the existence of steady motion, the dynamics is the same whether we take the motion of the air relatively to

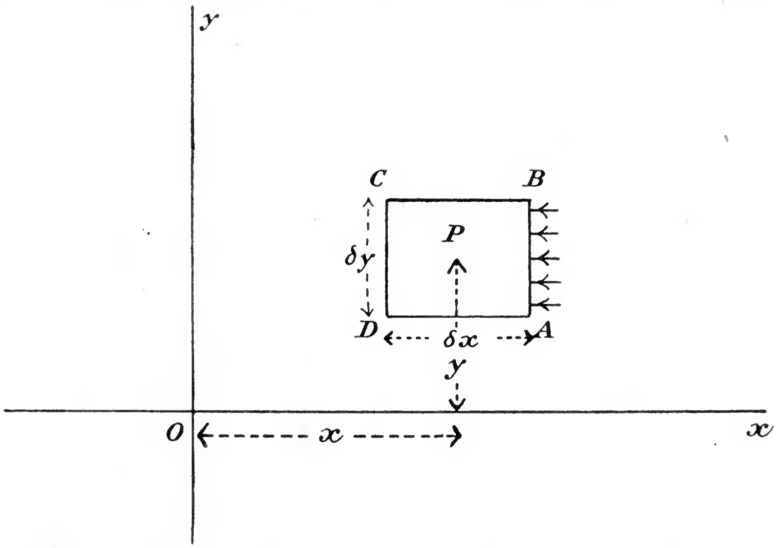

FIG. 70.—Section of Elementary Prism of Fluid.

the aeroplane, or whether we imagine the aeroplane at rest and the air streaming past it with the same (but oppositely directed) general velocity as the aeroplane is supposed to have. In experimental aerodynamics the second course is the more usual, a *wind channel* being used for the purpose. In the mathematical theory to be developed here we shall also adopt this course.

The important question whether steady motion such as we have defined can really exist is one that we shall not enter into here. There has been a certain amount of difference of opinion on the subject: we shall examine the mathematics on the assumption of the existence of steady motion, since the results obtained are, at any rate, not too widely divergent from experimental results.

142. **Forces Acting on a Drop of Fluid.**—Let us consider the two-dimensional steady motion of a non-viscous fluid. The absence of viscosity implies the important consequence that if we imagine any surface of separation in the fluid, the forces across this surface between the fluid

on one side and the fluid on the other act everywhere normally to the surface of separation. It also implies that where fluid is in contact with a solid boundary, whether moving or fixed, the effects of the fluid on the solid, and *vice versa*, are at every point of the surface of the boundary normal to this surface.

Take a drop of fluid defined by the small rectangle $ABCD$, whose centre, P, is at the point x, y, and whose sides have lengths δx, δy, Fig. 70. The drop is a prism on rectangular base and is of unit length. In two-dimensional motion we must of course assume the density of the fluid to be the same at all points in a line perpendicular to the planes of motion. We shall actually (in accordance with the assumption of § 140) take the density to be an absolute constant, ρ. Hence the mass of the narrow prism of fluid $ABCD$ is $\rho\,\delta x\,\delta y$.

The forces acting on this prism of fluid are twofold: forces, like gravity, coming from outside the fluid (leaving out mutual attractions of the parts of the fluid), which we call external forces, and forces due to the surface of separation which bounds the prism. It is a fundamental property of non-viscous fluid that the latter or surface forces over any element of surface of separation are not only normal to this element, but also independent of the direction of this normal. The force per unit area across an element of surface of separation at any point (x, y) is thus a function of (x, y) alone, and we call this the pressure—*i.e.* rate of pressure—at the point, denoted by p.

The motion of the prism of fluid $ABCD$ is defined by the forces acting on it. We omit the external forces, since air is so light that the internal pressures are far more potent than external forces which depend on the mass per unit volume. Hence the forces acting on the element $ABCD$ are obtained by considering the pressures over the four faces of the prism given by AB, BC, CD, DA (the forces on the rectangular faces at the ends of the prism balance one another because of the assumed two-dimensional nature of the motion).

Consider the pressure on the face AB. Since this face is plane all the elements of pressure are parallel, and to get the value of their total effect we simply add them up. But AB is taken very short. Hence this resultant is given if we multiply the length AB by the average value of the pressure over AB, *i.e.* by its value at the mid point of AB, co-ordinates $(x + \delta x/2, y)$. The pressure on the face AB is therefore

$$\delta y \,.\, p\left(x + \frac{\delta x}{2},\ y\right),$$

and it acts to the left. Similarly, the pressure on the face CD is

$$\delta y \,.\, p\left(x - \frac{\delta x}{2},\ y\right),$$

and it acts to the right. The effect of these pressures together is

$$\delta y\left\{p\left(x - \frac{\delta x}{2},\ y\right) - p\left(x + \frac{\delta x}{2},\ y\right)\right\}$$

to the right; the value is

$$\delta y\left\{p(x,\ y) - \frac{\delta x}{2}\frac{\partial p}{\partial x}\cdots - p(x,\ y) - \frac{\delta x}{2}\frac{\partial p}{\partial x}\cdots\right\},$$

which, neglecting higher powers of δx, becomes

$$-\delta x\,\delta y\,\frac{\partial p}{\partial x}$$

or

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \text{ per unit mass of fluid.}$$

This is the total effect parallel to the axis of x.

The total effect parallel to the axis of y is given by the faces BC, AD, and by a similar argument this is

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} \text{ per unit mass of fluid.}$$

As we take these to be the only forces acting on the prism $ABCD$, the equations of motion of the prism are

$$\left.\begin{aligned}\text{Acceleration along } x \text{ axis} &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \text{Acceleration along } y \text{ axis} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}.\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

143. **Equations of Motion.**—We have to discover the correct forms of these accelerations. Since the motion is taken to be steady, the velocity of a particle of fluid which is at the point (X, Y) at any moment is the same as that of another particle of fluid which happens to be at the point (X, Y) at some other moment. The velocity thus depends only on (X, Y), and we can say that if the components of this velocity parallel to the axes are u, v, then u, v are functions of (X, Y) only. If now a particle of fluid is at the point (X, Y) and then a moment Δt later is at the point $(X+\Delta X,\ Y+\Delta Y)$, the changes in its velocity components in time Δt are respectively

$$u(X+\Delta X,\ Y+\Delta Y)-u(X,\ Y),$$
$$v(X+\Delta X,\ Y+\Delta Y)-v(X,\ Y),$$

i.e. by Taylor's theorem

$$\Delta X\frac{\partial u}{\partial X}+\Delta Y\frac{\partial u}{\partial Y},\quad \Delta X\frac{\partial v}{\partial X}+\Delta Y\frac{\partial v}{\partial Y},$$

and the accelerations are

$$\frac{\Delta X}{\Delta t}\frac{\partial u}{\partial X}+\frac{\Delta Y}{\Delta t}\frac{\partial u}{\partial Y},\qquad \frac{\Delta X}{\Delta t}\frac{\partial v}{\partial X}+\frac{\Delta Y}{\Delta t}\frac{\partial v}{\partial Y},$$

which are

$$u\frac{\partial u}{\partial X}+v\frac{\partial u}{\partial Y},\quad u\frac{\partial v}{\partial X}+v\frac{\partial v}{\partial Y}. \quad \ldots\ldots\ldots \quad (2)$$

The particles of fluid in the prism $ABCD$ all have different velocities and accelerations. But if we take the rectangle $ABCD$ small enough we can use the average values, and so we say that the x, y accelerations to be used in equations (1) are the values at the point (x, y): we therefore get the following equations of motion :—

$$\left.\begin{aligned}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

144. **Equation of Continuity.**—As we are supposed to find the motion we must consider u, v as unknown functions of x, y. p is also some unknown function. Two equations are not sufficient to determine these functions—we must discover a third fact about the prism of fluid. This fact is that the prism of fluid, no matter how it changes in shape and position, must contain a constant amount of fluid.

After a time δt, let the rectangle $A_1B_1C_1D_1$ become $A_2B_2C_2D_2$, now

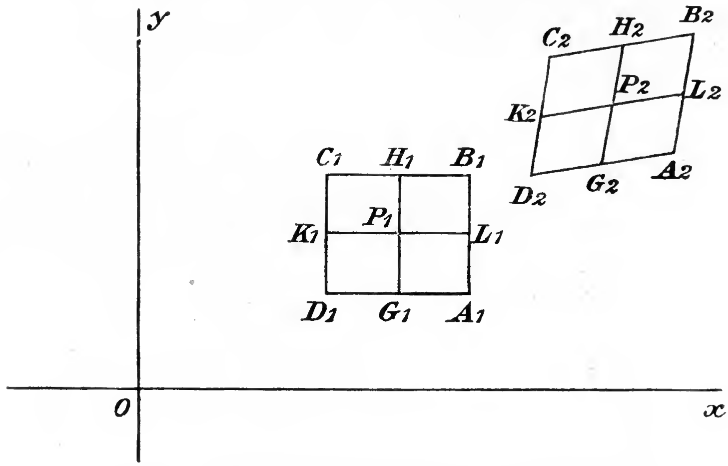

FIG. 71.—Motion and Distortion of Prism of Fluid.

no longer a rectangle, but, in general, a parallelogram (Fig. 71). The displacements of G_1, H_1 are

$$u\left(x,\ y-\frac{\delta y}{2}\right)\delta t,\ v\left(x,\ y-\frac{\delta y}{2}\right)\delta t,$$

$$u\left(x,\ y+\frac{\delta y}{2}\right)\delta t,\ v\left(x,\ y+\frac{\delta y}{2}\right)\delta t;$$

so that the x and y projections of G_2H_2 are respectively

$$\frac{\partial u}{\partial y}\delta y\ .\ \delta t,\ \delta y+\frac{\partial v}{\partial y}\delta y\ .\ \delta t.$$

Similarly, the x and y projections of K_2L_2 are respectively

$$\delta x+\frac{\partial u}{\partial x}\delta x\ .\ \delta t,\ \frac{\partial v}{\partial x}\delta x\ .\ \delta t.$$

The area of $A_2B_2C_2D_2$ can be put as equal to the product of the y projection of G_2H_2 into the x projection of K_2L_2, to the first order of small quantities, *i.e.*

$$\delta x\,\delta y\left\{1+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\delta t\right\}.$$

Thus we get the condition

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad (4)$$

the so-called **equation of continuity** for two-dimensional motion of an incompressible fluid.

We have then three equations of motion :

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0. \end{aligned}\right\} \qquad (5)$$

These equations are in general sufficient to determine the three unknown functions u, v, p. It is, as is usual in mathematical work of this sort, necessary to cast about for some means of obtaining these functions, either generally or in some specially selected cases. The most useful mathematical process when a number of functions are given by as many equations is to attempt to deduce an equation in which only one of the functions appears.

145. **Rotation of a Drop of Fluid.**—If we consider the first two of the equations (5), we at once think of the device of eliminating the function p by means of the fact $\frac{\partial}{\partial y}\frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\frac{\partial p}{\partial y}$. Since ρ is constant, we get

$$u\frac{\partial^2 u}{\partial x\,\partial y}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$
$$=u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y},$$

which at once simplifies (using the third equation (5)) into

$$u\left(\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2}\right)+v\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\,\partial y}\right)=0,$$

i.e.

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=0. \qquad (6)$$

Now this last equation is, physically speaking, the statement that the function $(\partial u/\partial y-\partial v/\partial x)$ is a constant quantity in all the history of the prism we are dealing with. The quantity must have a physical meaning, and we proceed to discover it.

Consider once again the conversion of the rectangle $A_1B_1C_1D_1$ into the parallelogram $A_2B_2C_2D_2$ (Fig. 71). It will be remembered that in getting the area of the latter we only used the x projection of one central dimension and the y projection of the other. If we now take the y projection of the first and the x projection of the other, we can at once assign a meaning to the function under consideration.

Take first the simpler case where the rectangle $A_1B_1C_1D_1$ is a square, so that $\delta x=\delta y$. The parallelogram $A_2B_2C_2D_2$ is now practically a rhombus : the y projection of K_2L_2 is $\frac{\partial v}{\partial x}\delta x\,.\,\delta t$, and the x projection of G_2H_2 is $\frac{\partial u}{\partial y}\delta y\,.\,\delta t$. This means that the angle between K_2L_2 and the x axis is $\partial v/\partial x\,.\,\delta t$, and the angle between G_2H_2 and the y axis is $\partial u/\partial y\,.\,\delta t$. But K_2L_2 and G_2H_2 make equal angles with the diagonal D_2B_2. It follows that the angle through which the diagonal D_1B_1 has *rotated* in the direction $x\rightarrow y$ is $\frac{1}{2}(\partial v/\partial x-\partial u/\partial y)\,\delta t$. The function $\frac{1}{2}(\partial v/\partial x-\partial u/\partial y)$ is

thus the angular velocity of the diagonal D_1B_1. The prism $ABCD$, therefore, goes through three transformations:

(i) its centre moves to a new position;
(ii) it is deformed into a rhombus;
(iii) it is rotated as a whole.

Using this new definition, we now see that the equation (6) means that the *angular velocity of rotation of the element*, or its *spin*, is *constant* for the element, no matter where it goes in its motion. We call the spin ζ (which is some function of x and y), and we have

$$u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=0. \qquad (7)$$

146. **Irrotational Motion, Bernoulli's Equation.**—As we are frankly seeking for simplifications in order to make solutions possible by mathematical analysis, we are tempted at this stage to continue this process by using an obvious special case suggested by (7). If we suppose ζ to be identically zero for any element, we are relieved of some complication in one of a set of three equations. As a matter of fact this simplification is usual in elementary hydrodynamics. We suppose that in the motion there is no angular velocity of any element. We therefore get the equations

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0,\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (8)$$

and as a third equation we can choose one or other of the first two of (5), or any equation involving p obtained from these two. We shall now show that there is a convenient form of this third equation.

Equations (5) give

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial x}$$
$$=\tfrac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)-2v\zeta.$$

Similarly, we get

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=\tfrac{1}{2}\frac{\partial}{\partial y}(u^2+v^2)+2u\zeta$$

If now ζ is taken to be zero, it is clear that we get

$$\frac{p}{\rho}+\tfrac{1}{2}(u^2+v^2)=\text{a constant} \qquad (9)$$

for all values of x, y, *i.e.* this function is the same at all points in the fluid. Putting q for the resultant velocity, we have

$$\frac{p}{\rho}+\tfrac{1}{2}q^2=\text{a constant}, \qquad (10)$$

a sort of equation of energy, but much more than an equation of energy. It shows not only that the kinetic energy of a given drop of fluid + a certain function is constant for the drop in all its wanderings, but also that this constant is the same for all drops per unit mass.

We have then the great simplification that in a fluid whose motion is *irrotational*, *i.e.* any drop has no spin but merely displacement and deformation, the equations of motion can be written

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0,\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\ \frac{p}{\rho}+\tfrac{1}{2}q^2=\text{constant}, \qquad (11)$$

a remarkably easy formulation of the dynamics. The first two equations define the velocity components, the third gives the pressure.

We could now easily obtain an equation for, say, u alone, by eliminating v between the first two equations (11). Since $\frac{\partial}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\frac{\partial v}{\partial y}$, we get

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=0;\\ \text{similarly,}\qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y}&=0.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (12)$$

Experience has shown, however, that it is advisable to retain the intimate connection between the velocity components exhibited by the equations (11), and to proceed mathematically in a manner far more suggestive of physical events.

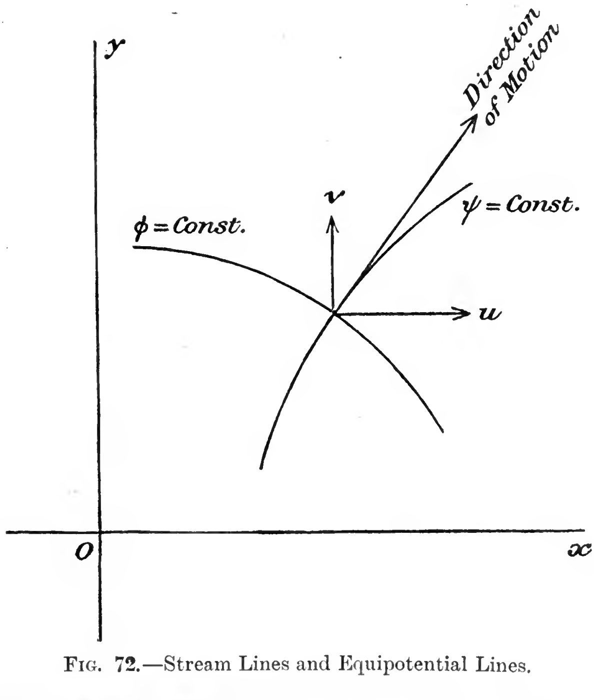

FIG. 72.—Stream Lines and Equipotential Lines.

147. **The Stream Function.**—The motion being steady, we can (as already remarked in § 141) draw a family of curves which are the stream lines, *i.e.* the fluid at a point on such a curve is actually moving along this curve; and any drop of fluid describes one of these curves during the motion. If a stream line has the equation

$$\psi(x, y) = \text{a constant}$$

(Fig. 72), then the direction of the tangent is given by

$$\frac{dy}{dx}=-\frac{\dfrac{\partial\psi}{\partial x}}{\dfrac{\partial\psi}{\partial y}},$$

and this ratio must be the same as v/u. Can we put $v = -\partial\psi/\partial x$, $u = \partial\psi/\partial y$? The answer is yes, because this is a necessary consequence of $\partial u/\partial x + \partial v/\partial y = 0$. If then we discover the form of $\psi(x, y)$ we have at once the stream lines, and also the velocity components at any point.

148. **The Potential Function.**—But a family of curves in a plane has a family of orthogonal curves, *i.e.* curves cutting the given family at right angles everywhere. Let the member of this new family that passes through any momentary position of a particle of fluid be

$$\phi(x, y) = \text{constant}.$$

Then the direction of the tangent is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}},$$

and this ratio must be the same as $-u/v$. Can we then write

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}?$$

Again the answer is in the affirmative, in consequence of $\partial v/\partial x - \partial u/\partial y = 0$.

We have then for ϕ and ψ the two equations

$$\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

—equations in every respect similar to those for u and v, and having the advantage of representing more clearly the actual motions of the fluid particles. The position of a particle is given by the co-ordinate ϕ on the stream line ψ, and the motion of the particle is along the stream line, with velocity given by

$$q = \left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\}^{\frac{1}{2}} = \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{\frac{1}{2}}.$$

The velocity component in any direction denoted by an element of length δs is $\partial\phi/\partial s$. If δs happens to be along a stream line, then $\partial\phi/\partial s$ is the resultant velocity at the point. In polar co-ordinates the velocity components along and perpendicular to the radius vector are $\partial\phi/\partial r$, $\partial\phi/r\partial\theta$.

ψ is called the *stream line function*, ϕ the *velocity potential* by analogy with the very similar mathematics in two-dimensional electrostatics.

Treat equations (13) in the same way as equations (11) were treated to obtain (12); two-dimensional irrotational hydrodynamics, or aerodynamics with no compression, can therefore be reduced to the equations

$$\left.\begin{array}{l} \nabla^2\phi = 0, \ \nabla^2\psi = 0, \ \dfrac{p}{\rho} + \dfrac{q^2}{2} = \text{a constant}, \\ \text{where } \nabla^2 \text{ is the operator} \\ \qquad \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

149. **Detailed Examination of Irrotational Motion.**—Having established the analytical formulation of irrotational motion, we must consider more closely the physical meaning of the lack of rotation. It is not an easy thing to conceive this, and we shall explain it in some detail.

Take a particle of fluid at P_1 at any moment, and let A_1 be a neighbouring particle at the same moment (Fig. 73). After a time interval δt, let P_1 have gone to P_2 and A_1 to A_2. Let us find the motion of A relatively to P.

P_1P_2 has x, y projections $u_P \,.\, \delta t$, $v_P \,.\, \delta t$; A_1A_2 has projections $u_A \,.\, \delta t$, $v_A \,.\, \delta t$; the notation being obvious. Thus the x, y projections of the displacement of A relatively to P are

$$(u_A - u_P)\,\delta t,\ (v_A - v_P)\,\delta t.$$

If P_1 has co-ordinates (x, y), and A_1 has co-ordinates δx, δy relatively to P_1, we have

$$u_A = u(x + \delta x,\ y + \delta y) = u(x,\ y) + \delta x\left(\frac{\partial u}{\partial x}\right)_P + \delta y\left(\frac{\partial u}{\partial y}\right)_P$$

$$v_A = v(x + \delta x,\ y + \delta y) = v(x,\ y) + \delta x\left(\frac{\partial v}{\partial x}\right)_P + \delta y\left(\frac{\partial v}{\partial y}\right)_P.$$

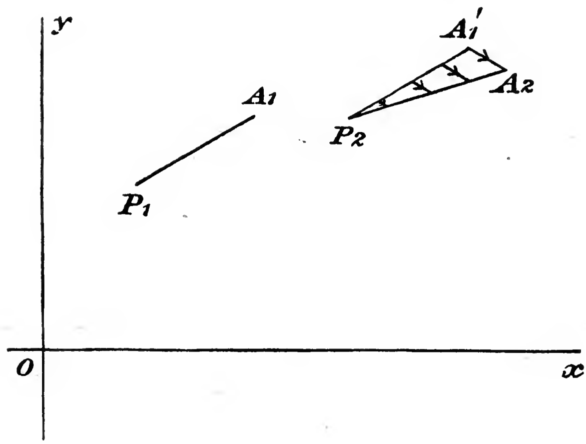

Fig. 73.—Relative Motion of Particles of Fluid.

Hence the rates of increase of the x, y projections of the displacement of A relatively to P are

$$\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y,\quad \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y,$$

the suffixes being now omitted for convenience. Remembering the analytical definition of irrotational motion, equations (11), these velocities are

$$a\delta x + h\delta y,\quad h\delta x - a\delta y, \qquad \qquad (15)$$

writing a, h for the quantities $\partial u/\partial x$, $\partial u/\partial y$, which are given if P_1 is given.

We have then a statement of the motion of the fluid relatively to P_1, true for all fluid in the vicinity. A particle at A_1 will have relative velocity components which have the same ratios for all particles on a line P_1A_1—in other words, all particles on a line through P_1 move, relatively to P_1, in parallel directions, but with velocities proportional to the distances from P_1. This is shown in Fig. 73, in which P_2A_1' is made equal and parallel to P_1A_1, so that $A_1'A_2$ represents the relative displacement of the

particle at A_1. We can then imagine the motion round P_1 to be defined by a number of rods through P_1, each one turning and expanding or contracting: this is true approximately for the fluid in the immediate vicinity, P_1 being any particle.

150. **Directions of Radial Relative Motion.**—The forms (15) at once suggest a simplification by a linear transformation, since a and h are really constants for the fluid round any chosen particle P_1. The transformation we require is given by the desirability of knowing which particles of fluid have only radial motion relatively to P_1, since there must be a sort of symmetry about the direction of the hypothetical rods on which these particles lie. We find these directions by making the relative velocity

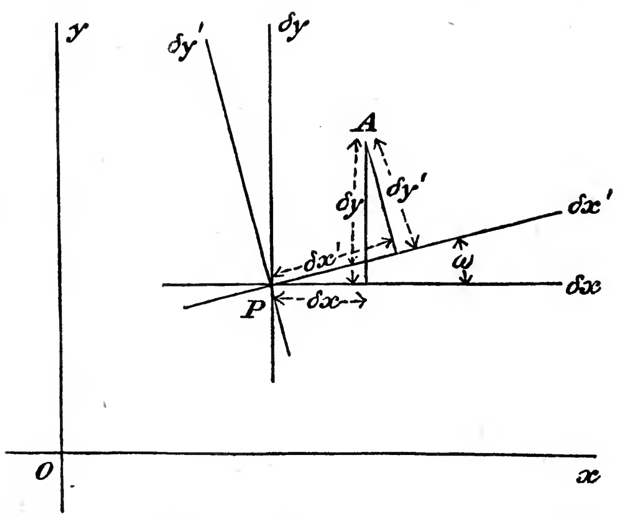

FIG. 74.—Relative Motion; Directions of Radial Relative Motion.

components proportional to the relative co-ordinates. The directions required are thus given by

$$\frac{a\,\delta x + h\,\delta y}{h\,\delta x - a\,\delta y} = \frac{\delta x}{\delta y},$$

so that

$$\delta y^2 + \frac{2a}{h}\,\delta x\,\delta y - \delta x^2 = 0. \qquad (16)$$

Thus there are two such directions (or rods) given by

$$\frac{\delta y}{\delta x} = \tan\omega, \ -\cot\omega, \qquad (17)$$

where ω is some angle dependent on the ratio a/h, *i.e.* on the motion and on the particle P_1 chosen, Fig. 74. The important thing to notice is that *these directions are mutually perpendicular.*

Transform to relative co-ordinates $\delta x'$, $\delta y'$ parallel to these directions. Then the component relative velocities must be

$$a'\,\delta x', \ b'\,\delta y',$$

because $\delta y' = 0$ must give resultant relative velocity parallel to $\delta x'$ and

vice versa. To find a', b' we say that the x, y components of $a'\,\delta x'$, $b'\,\delta y'$ are those given by (15). Hence

$$a'\delta x'\cos\omega - b'\delta y'\sin\omega \equiv a\,\delta x + h\,\delta y,$$

$$a'\delta x'\sin\omega + b'\delta y'\cos\omega \equiv h\,\delta x - a\,\delta y;$$

where

$$\delta x' = \delta x\cos\omega + \delta y\sin\omega,\quad \delta y' = -\,\delta x\sin\omega + \delta y\cos\omega.$$

Hence

$$(a'\cos^2\omega + b'\sin^2\omega)\delta x + \sin\omega\cos\omega\,(a' - b')\delta y \equiv a\,\delta x + h\,\delta y,$$

and

$$(a'\sin^2\omega + b'\cos^2\omega)\delta y + \sin\omega\cos\omega\,(a' - b')\delta x \equiv h\,\delta x - a\,\delta y.$$

Thus

$$a'\cos^2\omega + b'\sin^2\omega = a,\quad a'\sin^2\omega + b'\cos^2\omega = -\,a,$$

$$(a' - b')\sin\omega\cos\omega = h.$$

We get $a' + b' = 0$, so that

$$a' = \frac{h}{\sin 2\omega} = -\,b'.$$

Using equation (16), we find

$$a' = (a^2 + h^2)^{\frac{1}{2}} = -\,b'.$$

Hence the component relative velocities are

$$(a^2 + h^2)^{\frac{1}{2}}\,\delta x',\quad -\,(a^2 + h^2)^{\frac{1}{2}}\,\delta y'. \qquad (18)$$

151. **Irrotational Motion Referred to these Directions.**—If en we take four particles at A_1, A_2, A_3, A_4 which have at any moment

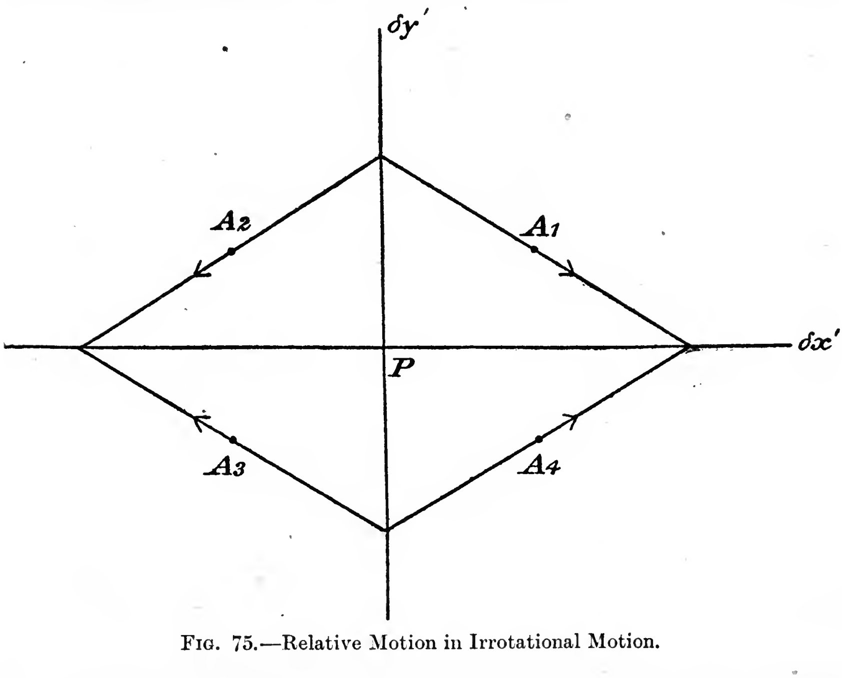

FIG. 75.—Relative Motion in Irrotational Motion.

the same numerical relative co-ordinates, the relative motions are along the sides of the rhombus, having these points for the middle points of its sides (Fig. 75). If we take a number of particles lying on any line at any

moment, the motions are as shown in Fig. 76 (*a*), so that a rhombus remains a rhombus with its diagonals along the same directions but of different lengths—one is longer than originally, the other is shorter.

A rectangle with sides parallel to $\delta x'$, $\delta y'$ will change as in Fig. 76 (*b*), *i.e.* it remains a rectangle and its sides have the old directions,

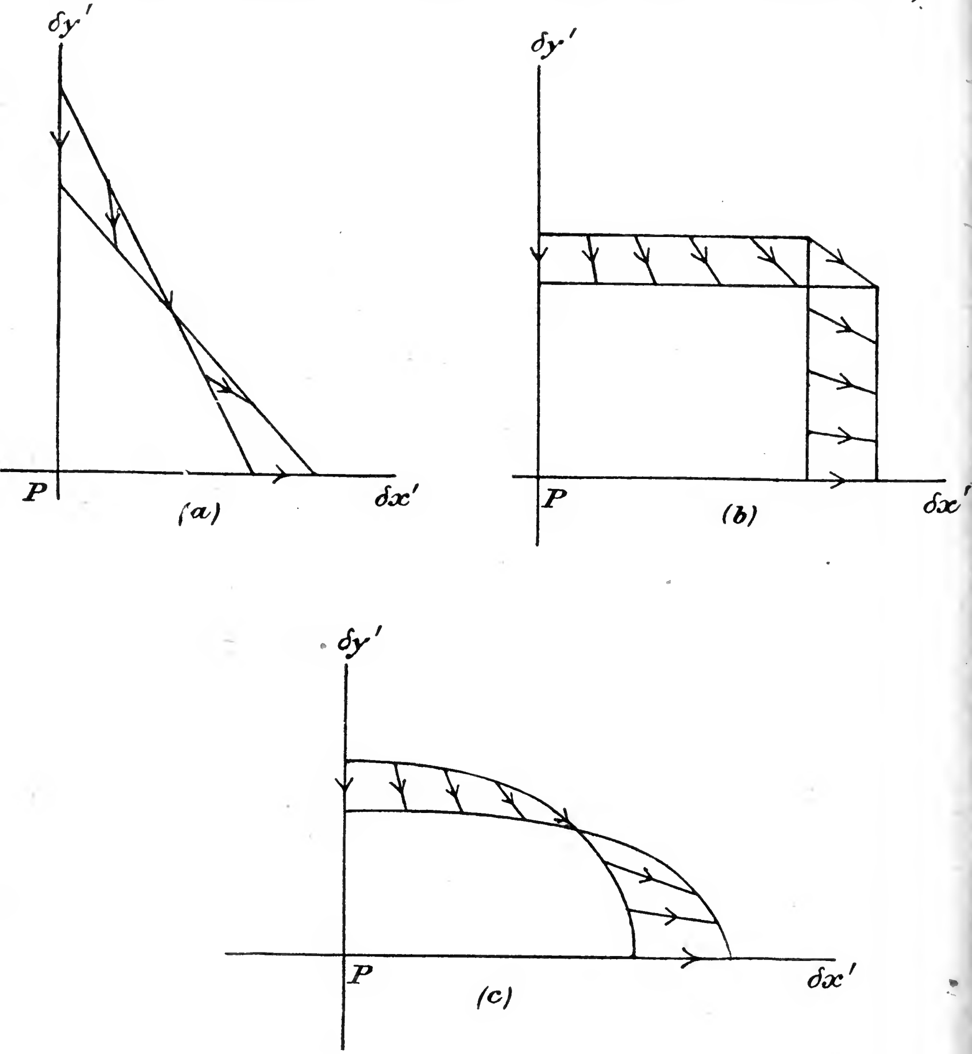

FIG. 76.—Relative Motion in Irrotational Motion.

but one pair is lengthened and the other shortened, the total area being unaltered.

More generally, an ellipse will be changed as in Fig. 76 (*c*).

This is irrotational motion: an element or drop of fluid gets deformed but does not rotate. It must be remembered, of course, that the drop as a whole is moving at the same time, but "whither the spirit is to go, it goes; it turns not when it goes." It must also be borne in mind

that at the next position of the drop the directions $\delta x'$, $\delta y'$ will be different in general, since these depend upon the ratio h/a, which is a function of x, y.

152. **Rotational Motion.**—All this is true only if we have $\partial u/\partial y = \partial v/\partial x$. If this is not the case, so that the x, y components of relative velocity are

$$a\delta x + h_1 \delta y, \quad h_2 \delta x - a\delta y, \quad \dots \dots \dots \quad (19)$$

where a is as before, but $h_1 = \partial u/\partial y$, $h_2 = \partial v/\partial x$, $h_1 \neq h_2$, then the directions of radial relative motion are given by

$$\frac{a\delta x + h_1\delta y}{h_2\delta x - a\delta y} = \frac{\delta x}{\delta y},$$

i.e.

$$h_1\delta y^2 + 2a\delta x\delta y - h_2\delta x^2 = 0, \quad \dots \dots \dots \quad (20)$$

so that the two directions for radial relative motion are not at right angles to one another. If we now refer the component relative velocities to these directions, we have again

$$a'\delta x', \quad b'\delta y',$$

but $\delta x'$, $\delta y'$ are measured along oblique axes. (The values of a', b' can be

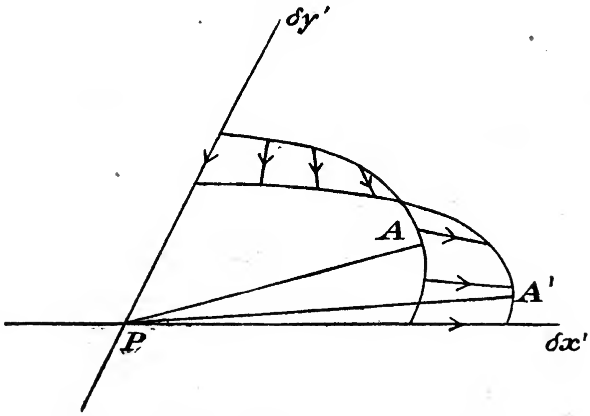

Fig. 77.—Relative Motion in Rotational Motion.

found, but we do not need them for the present argument.) There is symmetry in opposite quadrants about these axes, but the effect of the motion is to distort and rotate. In Fig. 77 an ellipse with its major axis along PA becomes an ellipse with its major axis along PA'.

153. **Examples of Irrotational Motion.**—One or two instances of irrotational motion will illustrate the above explanation. For irrotational motion we have $\nabla^2\psi = 0$. A simple case is where

$$\psi \equiv Ax,$$

A being a constant. This gives

$$u = 0, \quad v = -A,$$

so that the motion is uniform everywhere and the fluid moves as a solid

body with velocity A parallel to the negative direction of the y axis (Fig. 78). Clearly there is no rotation of any drop.

A similar result is obtained for $\psi \equiv By$, $\psi \equiv Ax + By$.

Consider the more advanced case

$$\psi \equiv Axy,$$

which also satisfies $\nabla^2\psi = 0$, A being a certain numerical constant. We get

$$u = Ax, \quad v = -Ay,$$

so that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = A, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0.$$

If then we choose any particle, the relative velocity components of the fluid near it are

$$A\,\delta x, \quad -A\,\delta y.$$

The history of a rectangular prism of fluid is thus as shown in Fig. 79, which gives the positions and shapes for equal intervals of time. At (a)

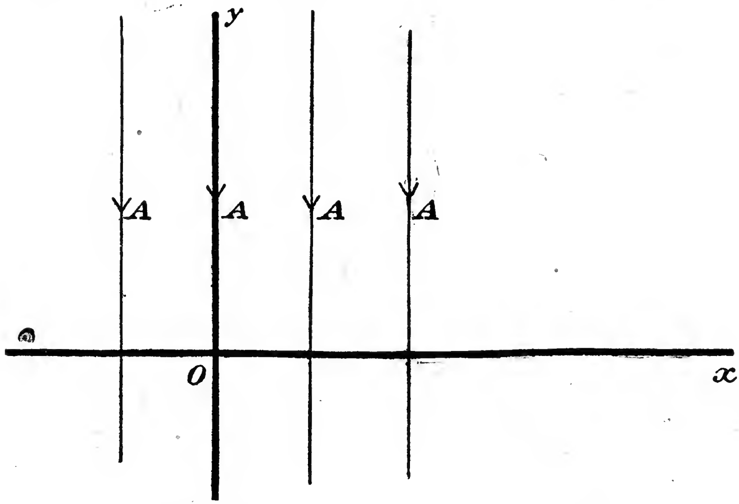

FIG. 78.—Obvious Case of Irrotational Motion.

the drop is supposed to be a square prism; at other points it is a rectangular prism. Note how the drop moves as a whole along the stream line for the middle point (in this example the stream lines are rectangular hyperbolas), whilst the shape changes in such a way that the volume (defined by the area in the plane x, y) is constant and there is no rotation. This example is easy because of the *constant directions* of the radial relative motions, indicated by $\delta x'$, $\delta y'$.

154. **Uniqueness of Solution.**—We now proceed to the solution of the general problem of two-dimensional irrotational motion of a non-viscous fluid. It is necessary to prove the following theorem :—

If a quantity of fluid is bounded by the surface of a solid body, so that the motion of the fluid must be tangential, and by any other surface (enclosing the former) over which the normal velocity is given at every point, then there is only one solution possible of the irrotational motion.

Let S, Fig. 80, be a boundary over which the normal velocity is given, S' a solid boundary so that the normal velocity is zero. The

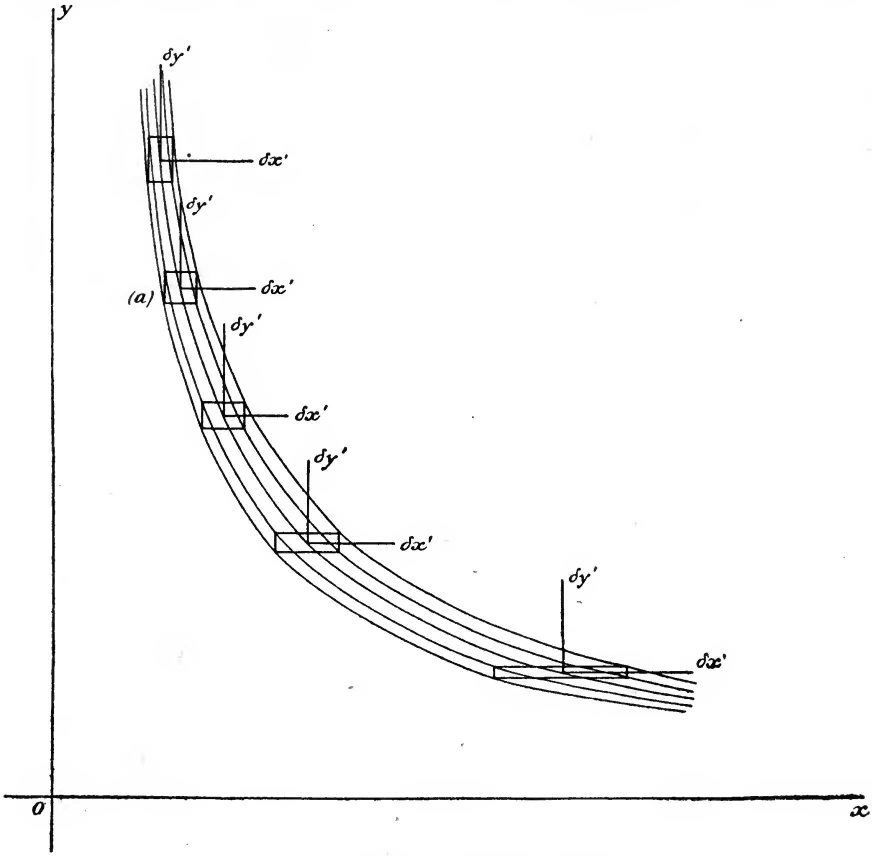

FIG. 79.—$\psi = Axy$:—Irrotational Motion.

motion is defined by the velocity potential ϕ, which in irrotational motion satisfies $\nabla^2\phi = 0$. If possible, let there be *two* solutions, ϕ_1, ϕ_2. Then $\partial\phi_1/\partial n$ is given over S, since the normal velocity is $-\partial\phi/\partial n$; also $\partial\phi_1/\partial n$ is zero over S'. Similarly, $\partial\phi_2/\partial n$ is given over S, and $\partial\phi_2/\partial n$ is zero over S'. It follows that the function $\phi_1 - \phi_2 \equiv \phi'$, which satisfies the equation $\nabla^2\phi' = 0$, has $\partial\phi'/\partial n$ zero over S and over S'.

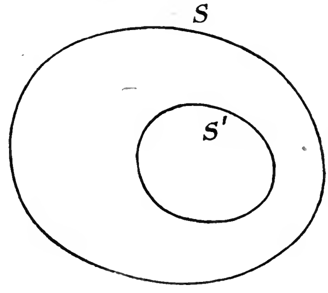

FIG. 80.

Thus ϕ' is an irrotational motion in which S, S' are effectively solid boundaries. Hence the corresponding stream lines must be such that any one is either a closed curve, or a curve that cuts itself, or a sort of spiral curve, Fig. 80. We shall show that these are all impossible in irrotational motion. To do this we introduce the *circulation* round a closed path.

155. **Circulation; Zero in Irrotational Motion.**—If we take any arc of a curve between two points A, B, then $u\dfrac{dx}{ds}+v\dfrac{dy}{ds}$ at any point is the velocity along the tangent to the curve, so that

$$\left(u\frac{dx}{ds}+v\frac{dy}{ds}\right)\delta s$$

is the total amount of fluid that *flows* between the ends of the element δs in unit time. Hence

$$\int_A^B (u\,dx+v\,dy)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

is the *flow* between A, B along the path AB in unit time. For a closed path like $ABCA$, Fig. 81 (a), the $\int_G (u\,dx+v\,dy)$ is called the circulation.

In irrotational motion the circulation is always zero. The area ABC can be split up into small rectangles, and if these are small enough it

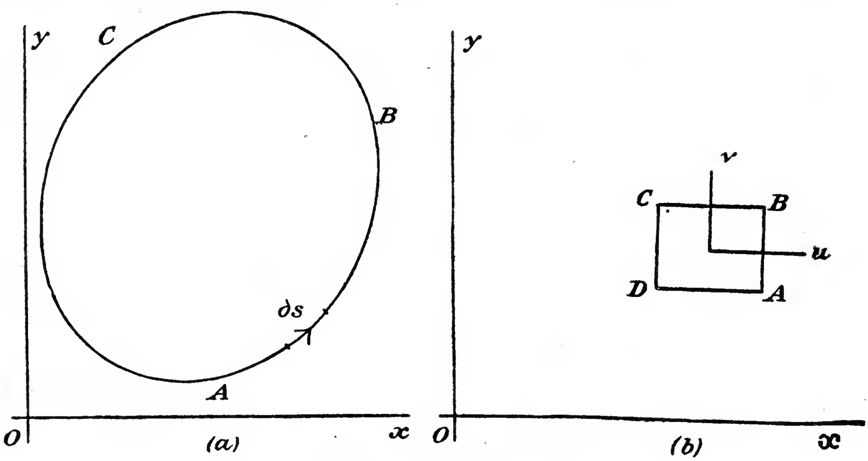

Fig. 81.—Circulation.

follows that the sums of the circulations round them is the same as the circulation round ABC, care being taken that the circulation is reckoned in the same sense for all the rectangles and for ABC, say anticlockwise.

Using the same notation as §§ 142, etc., the circulation round the rectangle $ABCD$ in Fig. 81 (b) is seen to be

$$\left\{v\left(x+\frac{\delta x}{2},\ y\right)-v\left(x-\frac{\delta x}{2},\ y\right)\right\}\delta y-\left\{u\left(x,\ y+\frac{\delta y}{2}\right)-u\left(x,\ y-\frac{\delta y}{2}\right)\right\}\delta x,$$

i.e.

$$\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\delta x\,\delta y,$$

which is zero for irrotational motion. Thus the circulation is zero for the small rectangle and therefore also for any closed path ABC, as in Fig. 81 (a).

But a stream line is such that at any point the velocity is tangential. Hence for any closed stream line we must have $\int q\,ds=0$, where q is the resultant velocity. A closed stream line is, therefore, possible only if the

velocity becomes zero and changes sign at two or any even number of points. The same applies to a stream line that turns in the form of a spiral. These possibilities can be dismissed in useful practical problems, except as affecting one or a finite number of stream lines: when *all* the stream lines are to have such singularities we have no natural problem. Hence in § 154 the quantity ϕ' must be identically zero, and $\phi_1 \equiv \phi_2$, which means that only one solution is possible.

The advantage of the theorem we have proved is at once obvious when we begin to attempt solutions of the equations of motion. With given boundary conditions we must somehow find the functions ϕ, ψ. A direct solution is rarely possible. We must, at best, *make a guess.* The theorem of § 154 shows that if a guess has been made and it satisfies the boundary conditions, then the guessed solution is unique, and therefore must be the solution we want.

156. **Solution of $\nabla^2\psi = 0$ in Polar Co-ordinates.**—As a guide in our guessing we can often use the following important theorem. The general solution of the equation $\nabla^2\psi = 0$ can be easily found in terms of polar co-ordinates. Let us use

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We readily find

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \quad \ldots \ldots \quad (22)$$

by straightforward substitution. Hence we have to solve the equation (22). Let us assume

$$\psi \equiv \Sigma A_n R_n \Theta_n,$$

where A_n is a number, R_n is some function of r, Θ_n of θ. If we substitute in (22) we find

$$\Sigma A_n \left\{\left(\frac{d^2R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr}\right)\Theta_n + \frac{R_n}{r^2}\frac{d^2\Theta_n}{d\theta^2}\right\} = 0. \quad \ldots \ldots \quad (23)$$

We assume that the term $A_nR_n\Theta_n$ by itself satisfies (22). We get that the quantity in each double bracket in the summation (23) must vanish. If now we write this in the form

$$\frac{1}{R_n}\left(r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr}\right) + \frac{1}{\Theta_n}\frac{d^2\Theta_n}{d\theta^2} = 0,$$

we have a function of r only + a function of θ only equals zero for all values of r and θ within the domain where the solution applies. This is possible only if each function is really a constant. Let us then write

$$\left.\begin{aligned} &\frac{d^2\Theta_n}{d\theta^2} + n^2\Theta_n = 0, \\ &r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - n^2R_n = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (24)$$

where n^2 is the constant to which we equate

$$\frac{1}{R_n}\left(r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr}\right),$$

and therefore also

$$-\frac{1}{\Theta_n}\frac{d^2\Theta_n}{d\theta^2}.$$

The suffix n attached to R_n, Θ_n refers to this quantity thus introduced. We find from (24) that

$$\Theta_n = \sin n\theta \text{ or } \cos n\theta,$$

$$R_n = r^n \text{ or } \frac{1}{r^n}.$$

It follows that each of the four quantities,

$$r^n \sin n\theta,\ r^n \cos n\theta,\ \frac{\sin n\theta}{r^n},\ \frac{\cos n\theta}{r^n}, \quad \ldots \ldots \quad (25)$$

where n has *any* value (which we may, of course, consider positive), is a solution of (22), and the general solution of $\nabla^2\psi = 0$ in polar co-ordinates is the sum of a number of such terms. Similar results hold for $\nabla^2\phi = 0$.

In practice we use integral values of n in the simpler types of problems, but there is no reason to restrict ourselves in this way in more difficult cases.

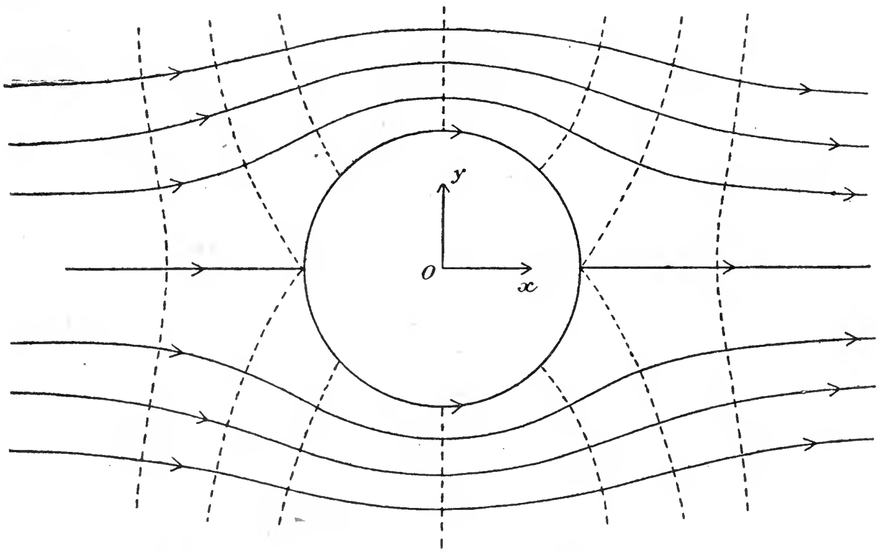

FIG. 82.—Irrotational Motion round Circular Cylinder.

157. **Example.**—As an illustration let us find the irrotational motion of fluid which extends to infinity but has to pass a circular cylinder whose axis is perpendicular to the plane of motion. We suppose that over the circle at infinity in the plane the velocity is a constant U in a constant direction. Over the cylinder the normal velocity is zero. By symmetry the circle (which is the trace of the cylinder in any plane of motion) and the extensions of the diameters must be stream lines. Hence, using the axis shown in Fig. 82, we must have, with polar co-ordinates, $\partial\psi/\partial r = 0$ over $\theta = 0$, and $\partial\psi/r\partial\theta = 0$ over the circle, $r = a$, say. We guess, by (25), $\psi = U(r - a^2/r)\sin\theta$, since $r \sin\theta$ and $\sin\theta/r$ each satisfies the equation $\nabla^2\psi = 0$, and we find the stream line we expect. We get

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta,$$

and the x, y velocity components are, in general,

$$\left\{1-\frac{(x^2-y^2)a^2}{(x^2+y^2)^2}\right\}U, \quad -\frac{2xya^2}{(x^2+y^2)^2}U.$$

When x, y are very large, these tend towards the values U, 0. Thus the boundary conditions are all satisfied and the solution suggested is the one required.

The pressure is given by

$$\frac{p}{\rho}+\frac{U^2}{2}\left\{1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right\}=\text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The stream lines are shown in Fig. 82. The ϕ curves, or *equipotential lines*, are the orthogonal curves (dashed).

158. The converse problem, where the fluid is at rest at infinity and any motion in it is caused by the uniform motion of the cylinder, is readily obtained. If the velocity of the cylinder is U backwards, so that the relative state is the same as before, we see that by impressing this velocity on all the fluid and on the body in the problem of § 157, we get the motion we require. Now the ψ function for the added motion is $-Uy$. Hence the ψ required is

$$\psi=U\sin\theta\left(-\frac{a^2}{r}+r\right)-Uy=-\frac{Ua^2\sin\theta}{r}, \quad \phi=\frac{Ua^2\cos\theta}{r}.$$

We find

$$u=-\frac{(x^2-y^2)a^2}{(x^2+y^2)^2}U, \quad v=-\frac{2xya^2}{(x^2+y^2)^2}U.$$

The co-ordinates are taken with origin on the axis of the moving cylinder.

We have zero velocity at infinity, and velocity components,

$$-U\cos 2\theta, \quad -U\sin 2\theta,$$

at any point θ on the moving cylinder. The normal component is

$$-U(\cos 2\theta\cos\theta+\sin 2\theta\sin\theta), \quad \textit{i.e.} \quad -U\cos\theta,$$

which is the normal velocity of the cylindrical boundary at the point θ. Relatively to the cylinder, the particles of fluid move in the directions of the stream lines in Fig. 82. If we imagine the figure to move to the left with the velocity U, at the same time as any stream line is described, we get the actual motion of any drop of fluid in space.

The second form of the problem differs from the first only as regards the absolute motion: the relative motion is the same in both cases, and the difference in pressure between any two points is the same in both. The consequence is that the resultant pressure on the cylinder is the same in both cases. As a matter of fact this is zero. For the numerical value of the velocity at any point θ is the same as at $\pi-\theta$ in both cases, so that the pressure distribution is, according to (26), symmetrical about the y axis.

159. **Solution in Cartesian Co-ordinates.**—We now return to the general equations of motion

$$\nabla^2\phi=0, \quad \nabla^2\psi=0, \quad \frac{p}{\rho}+\frac{q^2}{2}=\text{constant}.$$

It is difficult to discover functions ϕ, ψ for given boundary conditions. The converse problem is very much easier: we assume values of ϕ, ψ, and deduce the corresponding boundary conditions. We thus have *a* solution satisfying these conditions, and it must be *the* solution, since it is unique.

As the two equations $\nabla^2\phi = 0$, $\nabla^2\psi = 0$ are the same for all such problems as we are now considering, we shall write down an *a priori* general statement about *all* potential and stream-line functions in irrotational two-dimensional motion of a non-viscous fluid.

The operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

can be split up into two linear differential operators

$$\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right), \quad \ldots \ldots \ldots \quad (27)$$

where $\iota^2 = -1$, and therefore the function ϕ, say, satisfies the equation

$$\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\phi = 0.$$

Putting

$$\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\phi = \phi',$$

we get

$$\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\phi' = 0,$$

so that ϕ' is some function of $(x + \iota y)$, say $f(x + \iota y)$. Hence

$$\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\phi = f(x + \iota y).$$

But $\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)$, operating on any function of $x + \iota y$, gives another function of $(x + \iota y)$. Hence

$$\phi = F_1(x + \iota y) + F_2(x - \iota y), \quad \ldots \ldots \ldots \quad (28)$$

in which

$$\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)F_1(x + \iota y) = f(x + \iota y)$$

is the definition of F_1, and F_2 is an arbitrary function. Thus we have ϕ as the sum of two arbitrary functions of $x - \iota y$, $x + \iota y$ respectively. In the same way,

$$\psi = F_3(x + \iota y) + F_4(x - \iota y), \quad \ldots \ldots \ldots \quad (29)$$

where F_3, F_4 are also two arbitrary functional forms. But in reality the four forms F_1, F_2, F_3, F_4 are not absolutely arbitrary. First, we must have ϕ and ψ both real; and secondly, we have the relations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

The second set of conditions give

$$F_1'(x + \iota y) + F_2'(x - \iota y) \equiv \iota F_3'(x + \iota y) - \iota F_4'(x - \iota y),$$
$$\iota F_1'(x + \iota y) - \iota F_2'(x - \iota y) \equiv -F_3'(x + \iota y) - F_4'(x - \iota y),$$

where $F_1'(x + \iota y)$ means $\dfrac{d}{d(x + \iota y)}F_1(x + \iota y)$, etc. Hence, dividing the second equation by ι and comparing with the first, we get

$$F_1'(x + \iota y) = \iota F_3'(x + \iota y), \quad F_2'(x - \iota y) = -\iota F_4'(x - \iota y),$$

so that

$$\left.\begin{aligned} \phi &= F_1(x + \iota y) + F_2(x - \iota y), \\ \iota\psi &= F_1(x + \iota y) - F_2(x - \iota y), \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (30)$$

leaving out arbitrary constants, since these do not affect the physical aspect of the problem, in which we only need the actual ϕ and ψ curves plotted, and the differential coefficients of ϕ and ψ. The first condition, viz. that ϕ and ψ are real, gives

so that
$$\left.\begin{aligned}\phi &= \frac{F(x+\iota y)+F(x-\iota y)}{2}\\ \psi &= \frac{F(x+\iota y)-F(x-\iota y)}{2\iota}\end{aligned}\right\} \text{ or } \left\{\begin{aligned}\phi &= \frac{F(x+\iota y)-F(x-\iota y)}{2\iota}\\ &\text{and}\\ \psi &= \frac{-F(x+\iota y)-F(x-\iota y)}{2}\end{aligned}\right.$$

where F is some real functional form. The two alternatives give

or
$$\left.\begin{aligned}\phi+\iota\psi &= F(x+\iota y), \quad \phi-\iota\psi = F(x-\iota y),\\ \psi+\iota\phi &= -F(x-\iota y), \quad \psi-\iota\phi = -F(x+\iota y).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (31)$$

We thus see that the general solution is obtained by taking $(\phi + \iota\psi)$ to be some function of $(x + \iota y)$ and equating real and imaginary parts, or by taking $(\psi + \iota\phi)$ to be some function of $(y + \iota x)$ and equating real and

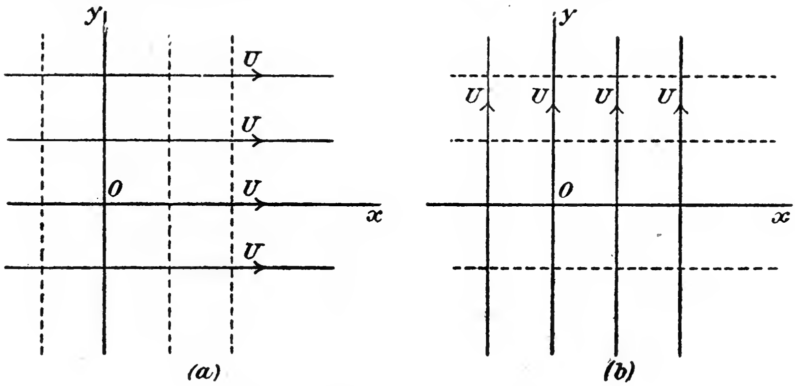

Fig. 83.—Two Solutions for $w \equiv z$.

imaginary parts. This means that if we assume some functional relation between $(\phi + \iota\psi)$ and $(x + \iota y)$, we can either treat ϕ as the potential function and ψ as the stream-line function, or ϕ as the stream-line function and ψ as the potential function, the plane of x, y being turned through 90°. Putting $z \equiv x + \iota y$, $w \equiv \phi + \iota\psi$, and assuming some relation $w = w(z)$, it follows that this relation defines *two* solutions of hydrodynamical problems.

160. *E.g.* take $w(z) \equiv Uz$; then $\phi = Ux$, $\psi = Uy$, or $\psi = -Ux$, $\phi_1 = Uy$, the motions being as shown in Fig. 83 (*a*), (*b*) respectively.

Or, suppose we use

$$w(z) \equiv \frac{a}{z} \equiv \frac{a}{x+\iota y} = \frac{a(x-\iota y)}{x^2+y^2}.$$

Then either

$$\text{(i)} \quad \phi = \frac{ax}{x^2+y^2} = \frac{a\cos\theta}{r}, \quad \psi = -\frac{ay}{x^2+y^2} = -\frac{a\sin\theta}{r},$$

giving the velocities radially and transversely as

$$-\frac{a\cos\theta}{r^2}, \quad -\frac{a\sin\theta}{r^2}.$$

the motion being as in Fig. 84 (a); or

$$\text{(ii)}\ \psi = -\frac{a\cos\theta}{r},\quad \phi = -\frac{a\sin\theta}{r},$$

giving radial and transverse velocities

$$\frac{a\sin\theta}{r^2},\quad -\frac{a\cos\theta}{r^2},$$

the motion being as in Fig. 84 (b). (The arrows should be reversed.)

In both the cases just given the two alternatives do not give actually

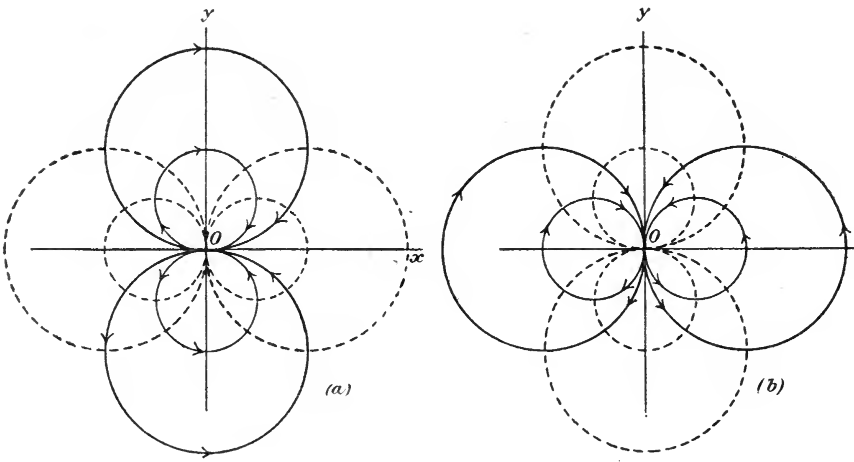

FIG. 84.—Two Solutions for $w = \dfrac{a}{z}$.

different problems, because it happens that the ϕ and ψ curves in each are really similar curves.

But if we take

$$w(z) \equiv U\log_e z \equiv U\log_e(x+\iota y) = U\log_e(r+\iota\theta) = U\log_e r + \iota U\theta,$$

we get either

$$\text{(i)}\ \phi = U\log_e r,\quad \psi = U\theta,$$

giving radial and transverse velocities

$$\frac{U}{r},\ 0,$$

the motion being as in Fig. 85 (a); or

$$\text{(ii)}\ \psi = -U\log_e r,\quad \phi = U\theta,$$

giving radial and transverse velocities

$$0,\ \frac{U}{r},$$

with motion as in Fig. 85 (b). Such a case is called **cyclic motion.**

The student may perhaps be at a loss to understand how the motion in Fig. 85 (*b*) can be irrotational, considering that each particle of the fluid goes round in a circle. He will notice that the velocity round any circle varies inversely as the radius, so that particles on different circles get round in different times, which vary as the radius squared; and this is just the condition that two particles near to one another on the same radius at any moment should for a short interval of time remain on a line parallel to their original join. In fact, if a drop is taken with its centre on any circular stream line, the instantaneous directions of radial relative motion bisect the angles between the radius and tangent at the point, and are therefore at right angles. Here the directions of radial relative motion vary from time to time: in the easy cases above, § 153, these directions were fixed. But it should be remembered that these directions must be normal to one another in irrotational motion.

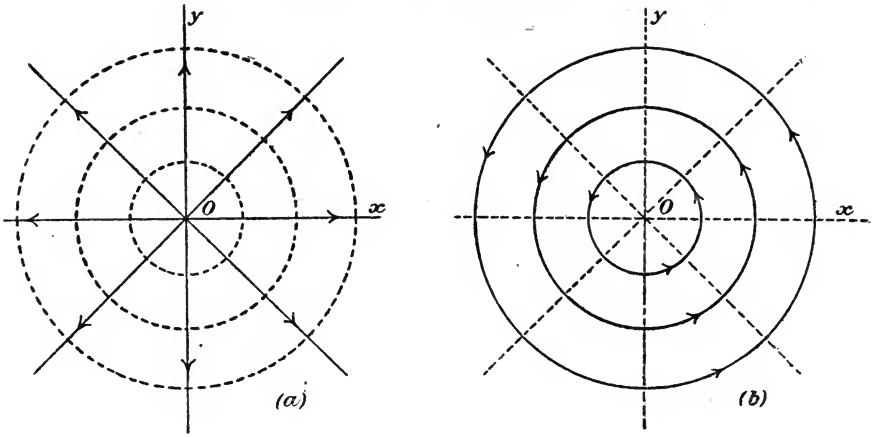

FIG. 85.—Two Solutions for $w \equiv U \log_e z$.

In the case of the cylinder of radius a with fluid passing it with velocity U, we have

$$w = U\left(z + \frac{a^2}{z}\right),$$

so that

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta,\quad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta.$$

The alternative problem with

$$\phi = U\left(r - \frac{a^2}{r}\right)\sin\theta,\quad \psi = -U\left(r + \frac{a^2}{r}\right)\cos\theta,$$

admits of a physical interpretation, but it is rather artificial.

For any given form of $w(z)$ we choose one or the other alternative, according to the nature of w and the sort of solution we seek.

161. **Flow through a Gap.**—Let us now try the following problem. Imagine a solid boundary of the form given by the hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{-b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

extended into a cylinder perpendicular to the x, y plane, and the fluid to flow irrotationally through the opening formed between the two parts of the cylinder.

It seems reasonable to suppose that the stream lines in any x, y plane are the confocal hyperbolas, since the ϕ curves must cut the hyperbola of the given boundary normally everywhere, and this suggests confocal ellipses.

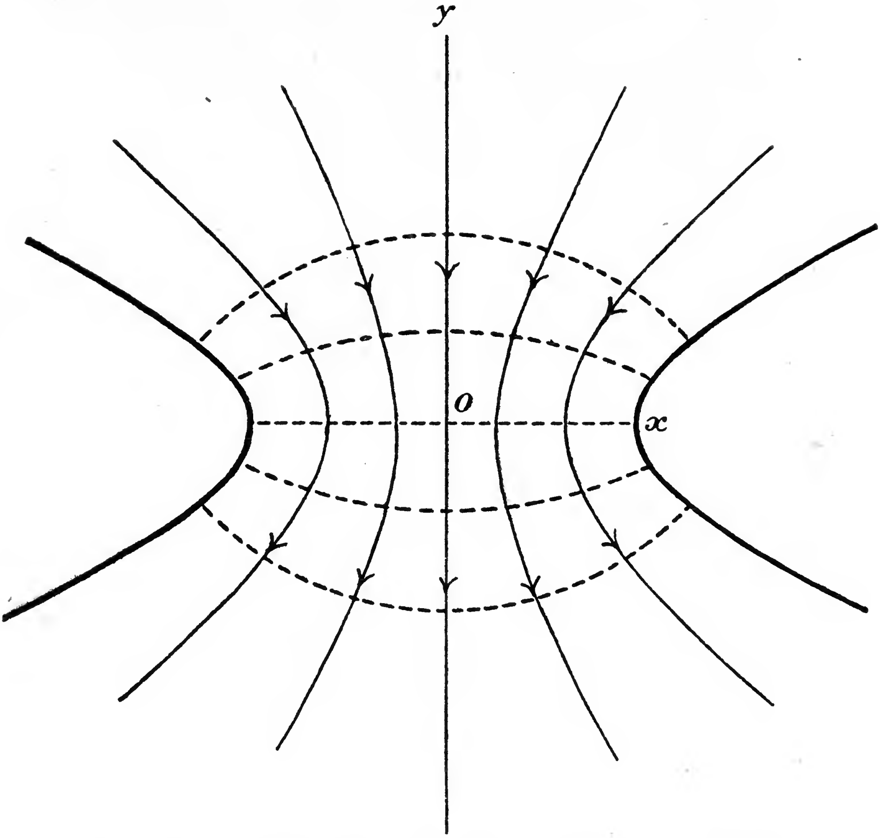

FIG. 86.—Irrotational Motion through a Gap defined by a Hyperbola.

Let, then, the ϕ and ψ families of curves be the ellipses

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{-b^2+\lambda}=1, \quad \lambda=b^2 \text{ to } \lambda=\infty,$$

and the hyperbolas

$$\frac{x^2}{a^2-\mu}+\frac{y^2}{-b^2-\mu}=1, \quad \mu=0 \text{ to } \mu=a^2.$$

We therefore write

$$x=(a^2+\lambda)^{\frac{1}{2}}\cos\left(\frac{\psi}{U}\right), \quad y=(\lambda-b^2)^{\frac{1}{2}}\sin\left(\frac{\psi}{U}\right),$$

$$x=(a^2-\mu)^{\frac{1}{2}}\cosh\left(\frac{\phi}{U}\right), \quad y=(b^2+\mu)^{\frac{1}{2}}\sinh\left(\frac{\phi}{U}\right),$$

suggesting

$$x=c\cosh\frac{\phi}{U}\cos\frac{\psi}{U}, \quad y=c\sinh\frac{\phi}{U}\sin\frac{\psi}{U}, \qquad (33)$$

where c, U are some constants. If this is to be true, we must have

$$(a^2+\lambda)^{\frac{1}{2}} = c\cosh\frac{\phi}{U},\quad (a^2-\mu)^{\frac{1}{2}} = c\cos\frac{\psi}{U}$$

$$(b^2+\mu)^{\frac{1}{2}} = c\sin\frac{\psi}{U},\quad (\lambda-b^2)^{\frac{1}{2}} = c\sinh\frac{\phi}{U}.$$

All is right if we use $c^2 = a^2 + b^2$, so that c is half the distance between the foci. We thus have the equations (33), which give

$$z = c\cosh\frac{w}{U} \text{ or } w = U\cosh^{-1}\left(\frac{z}{c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The stream lines are the hyperbolas

$$\frac{x^2}{c^2\cos^2\dfrac{\psi}{U}} - \frac{y^2}{c^2\sin^2\dfrac{\psi}{U}} = 1,$$

and the equipotential curves are the ellipses

$$\frac{x^2}{c^2\cosh^2\dfrac{\phi}{U}} + \frac{y^2}{c^2\sinh^2\dfrac{\phi}{U}} = 1.$$

All the conditions of the problem are satisfied if we get zero velocity at infinity, and the given velocity somewhere, say at the origin of co-ordinates. We have to deduce the velocity, if possible, without first evaluating ϕ or ψ in terms of x, y. This is done by means of the following general argument.

162. **The Function** $\zeta \equiv dz/dw$.—We have

$$\frac{dw}{dz} = \frac{d(\phi+\iota\psi)}{d(x+\iota y)} = \frac{\partial\phi}{\partial x} + \iota\frac{\partial\psi}{\partial x} = u - \iota v.$$

Now dw/dz is some function of z, *i.e.* of $x + \iota y$, so that it can be split up into a real and into an imaginary part: the real part is $\partial\phi/\partial x$, and the imaginary part is $\partial\psi/\partial x$. Hence the *modulus* of dw/dz is

$$\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2\right\}^{\frac{1}{2}},$$

i.e. the velocity q. Further, the *argument* of dw/dz is

$$\tan^{-1}\left(\frac{\dfrac{\partial\psi}{\partial x}}{\dfrac{\partial\phi}{\partial x}}\right) = \tan^{-1}\left(\frac{-v}{u}\right),$$

so that the argument represents the direction of motion with the v component reversed. To obtain the correct direction of motion we consider dz/dw, which we call ζ. We have

$$\zeta = \frac{dz}{dw} = \frac{1}{u-\iota v} = \frac{u+\iota v}{u^2+v^2} = \frac{1}{q}\left(\frac{u}{q} + \iota\frac{v}{q}\right).$$

Hence the arithmetical value of the velocity is given by the inverse of the modulus of ζ; the direction of this motion is defined by the argument of ζ.

163. Returning to the problem of § 161, we have for the velocity

$$U\left|\frac{1}{\sqrt{z^2-c^2}}\right|,$$

so that at the origin the velocity is U/c.

If, then, we are to have prescribed velocity V at the origin, we get

$$z = c\cosh\left(\frac{w}{cV}\right). \qquad (35)$$

The stream lines are the hyperbolas

$$\frac{x^2}{c^2\cos^2\frac{\psi}{cV}} - \frac{y^2}{c^2\sin^2\frac{\psi}{cV}} = 1, \qquad (36)$$

and the equipotential curves are the ellipses

$$\frac{x^2}{c^2\cosh^2\frac{\phi}{cV}} + \frac{y^2}{c^2\sinh^2\frac{\phi}{cV}} = 1. \qquad (37)$$

The resultant velocity at any point is

$$V\left|\frac{c}{\sqrt{z^2-c^2}}\right|.$$

The pressure is given by

$$\frac{p}{\rho} + \frac{V^2c^2}{2}\left|\frac{1}{\sqrt{z^2-c^2}}\right|^2 = \text{a constant},$$

i.e.

$$\frac{p}{\rho} + \frac{V^2c^2}{2\,|z^2-c^2|} = \text{a constant},$$

since the modulus of the product $\sqrt{z^2-c^2}\ \sqrt{z^2-c^2}$ is the same thing as the product of the moduli, *i.e.* as the square of the modulus of $\sqrt{z^2-c^2}$. The pressure is given by

$$\frac{p}{\rho} + \frac{V^2c^2}{2\{(x^2-y^2-c^2)^2+4x^2y^2\}^{\frac{1}{2}}} = \frac{C}{\rho}, \text{ a constant}. \qquad (38)$$

It is clear that the value of C must be greater than the greatest value of the second term on the left-hand side, since p must be positive everywhere. It is readily seen that for any value of x, this second term is greatest for $y=0$. Hence we can say that $C > \dfrac{V^2c^2\rho}{2(x^2-c^2)}$ for all values of x admissible with $y=0$, *i.e.* across the opening in the cylinder. Taking, *e.g.*, one of the verticles of the hyperbola, we have $x=a$, and then

$$C > \frac{V^2c^2\rho}{2(a^2-c^2)}. \qquad (39)$$

If, then, the solution we have just written down is to be possible, we must have C greater than a quantity which is determined by the *shape* of the hyperbola which defines the opening. C is, in fact, the requisite pressure where the velocity is zero, *i.e.* at infinity. *But for practical problems C must be finite,* otherwise we get the pressure infinite everywhere. Hence it is not allowable to use the above solution if a approaches the value c indefinitely. *We cannot suppose the fluid to bend right round a sharp corner.*

CHAPTER VI

DISCONTINUOUS FLUID MOTION: FREE STREAM LINES

164. We have in Chapter V. given a brief account of the theory of hydrodynamics as applicable to our subject. The conclusion reached at the end of the chapter suggests that we have to examine our assumptions and methods more closely. We have to consider two important questions: Can we rely on pressures being dependent only on relative motion? Can we really accept the statement that there is no resultant pressure on the cylinder in § 157? The first question is important because in theoretical and practical investigations it is convenient to use sometimes one form of the motion, viz. the body moving through the fluid "at rest," and sometimes the other form, viz. the fluid moving past the body. It is clear *a priori* that the two methods would not give the same results if we considered the motion as being set up in each case, *i.e.* in the first the body is made to acquire its velocity, in the second the fluid is somehow endowed with its motion. If, however, we are considering steady motion, in which the velocities are supposed already in existence and merely maintained, the time element disappears and only the relativity of the motions affects the total pressure on the body.

The second question cannot be dismissed in such a manner. Experience tells us that bodies moving relatively to fluids are subject to a resultant pressure in the nature of a resistance, and we must investigate how such a pressure can be accounted for. If we refer back to the case of the cylinder in moving fluid, we see that we there assumed that the fluid flows all round the cylinder. In practice it is found that the cylinder acts as a sort of screen, so that behind it there is a region in which the motion of the fluid is quite different from the irrotational motion investigated in § 157. Experiments made at the National Physical Laboratory show that there is rotational or vortex motion set up, and of a complicated type. It would seem that it is essential to consider the problem of rotational motion.

165. **Discontinuous Fluid Motion.**—Yet it is not quite essential to introduce rotation at the present stage. When water flows out of a hole in a vessel it is possible to get almost irrotational motion in a jet, which means that the fluid does not spread all round the obstacle. When the wind blows over the glass screen of a motor-car, the person seated behind it feels little, if any, of the effect of the wind. We therefore proceed to examine whether, and how, irrotational motion can exist ir which a solid acts as a screen.

It is difficult to conceive this for a fluid like air; with water the

phenomenon of screening is very familiar. The jet already mentioned, a stream flowing from a tap, etc., are everyday instances. As far as the water is concerned, it is not far wrong to consider the surrounding air as not sharing in the motion at all. The free surface of the water is thus a *surface of discontinuity*, where the motion of the water suddenly disappears.

If we imagine the same to happen in the case of air, we have a problem yielding a resultant air pressure due to relative motion of a solid in air. This is only a rough approximation: the space screened by the body is actually filled with air in a state of turbulence, not only in the case of a cylinder, but also in the case of a jet from a vessel, or air past a plane or any other obstacle. (See *Report of Advisory Committee for Aeronautics*, 1913–14.) We shall, nevertheless, examine the problems with assumed surfaces of discontinuity, as the results have been found to give tolerable agreement with experimental fact. The analytical processes and arguments are of considerable difficulty, and in what follows, only the simplest cases are considered.

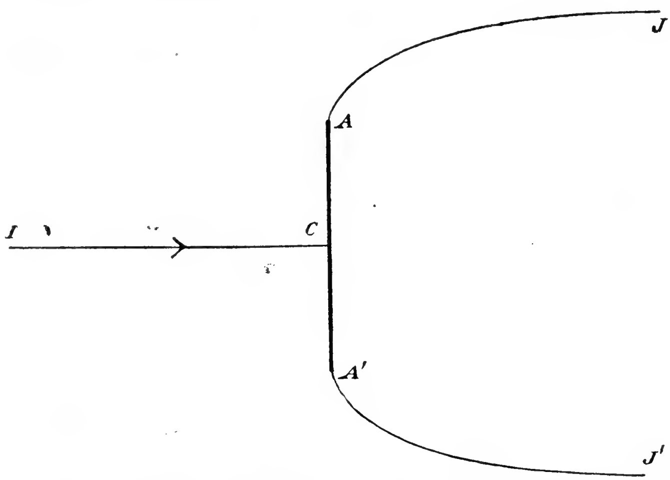

FIG. 87.—Discontinuous Motion: AJ, $A'J'$ are Free Stream Lines.

166. **Boundary Conditions.**—We must first find the form of the boundary conditions. The boundary may consist of the surfaces of solids, free boundaries on the other side of which there is either no fluid at all, or, at any rate, no motion, and the surface at infinity. For the surfaces of solids the normal relative velocity of the fluid must be zero. Over the surface at infinity the velocity and direction will be supposed given. *Over any free boundary the velocity must be constant*, since the equation $p/\rho + \frac{1}{2}q^2 =$ constant, holds right up to the free boundary, where we have for p the constant pressure beyond.

Although it is true that our method will be the converse one of more or less guessing solutions, yet we need some guide to the sort of solution desirable; hence it is an advantage to have in front of us a graphical statement of the boundary conditions. These conditions refer to the velocity of a particle of fluid at the boundaries, and we therefore express them by means of the function ζ, already defined.

167. **Graphical Representation: The ζ Plane.**—Let us begin with the type of problem that is fundamental to aeroplane motion, viz. the normal flow of a fluid past a flat plate, the motion being steady, irrotational, and in two dimensions. It is clear that the motion of the fluid in any x, y plane is symmetrical about the perpendicular bisector of the trace of the plate in this plane. Let ACA', Fig. 87, be the trace, IC the stream line along the line of symmetry. Suppose that AJ, $A'J'$ are free stream lines.

We can split this problem into two equal parts by supposing that IC

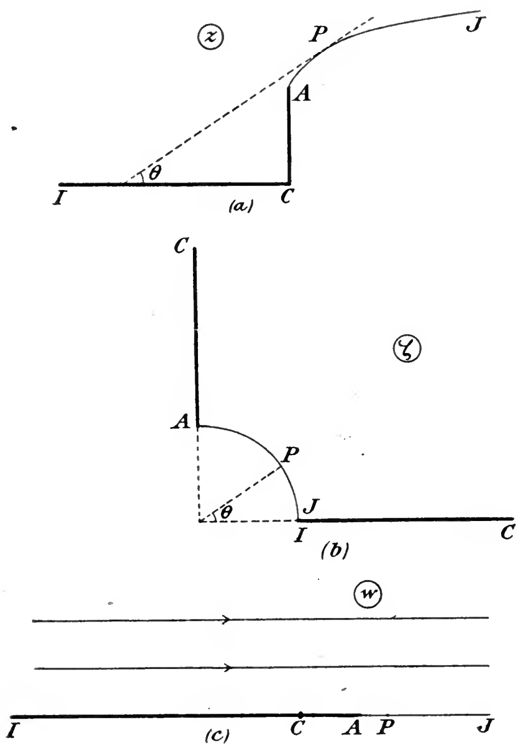

Fig. 88.—z, ζ, and w Planes for Problem of Fig. 87, divided into Equal Portions by a Plane through IC.

is a boundary defined by the surface of a solid. Fig. 88 (a) is the z plane of this problem. Let the velocity be unity at infinity, of course parallel to IC: this is the boundary condition for infinity. For the boundary ICA we want zero normal velocity. For the free boundary AJ we want constant velocity, and this must be unity since AJ goes to infinity: further, the direction of AJ must be ultimately parallel to IC.

Now, although we do not yet know the form of the line AJ, so that in a sense the boundary conditions are not yet given, we can by means of the function ζ obtain a *definite* statement which will enable us to effect a solution. This is, in fact, the merit of the method we are now developing.

Plot the curve which represents the boundary conditions. This has a meaning, for we have $\zeta = \frac{1}{q}\left(\frac{u}{q} + \iota\frac{v}{q}\right)$, so that, if we take a point in the "ζ plane" considered as an Argand diagram, we use the co-ordinates u/q^2, v/q^2 in Cartesians, or $1/q$, $\tan^{-1} v/u$ in polars. For any straight *given* boundary the direction $\tan^{-1} v/u$ is a constant, and in the ζ plane we get a straight line through the origin; for the free boundary $1/q$ is given as equal to 1, and this gives the arc of a circle of unit radius round the origin in the ζ plane; finally, at infinity the velocity is unity in a given direction, viz. parallel to the line IC, so that in the Argand ζ diagram the corresponding point is (1, 0). Hence for the case in Fig. 88 (a) we get the ζ curve in Fig. 88 (b). In all our figures rigid boundaries will be indicated by bold lines.

From I to C in the z plane, Fig. 88 (a), $v = 0$, whilst at I we have in addition $q = 1$; at C we have $q = 0$, since at C the ψ curve has a singular point for which $\partial\psi/\partial x = \partial\psi/\partial y = 0$; hence we get the line IC in the ζ plane, Fig. 88 (b). From C to A in the z plane we have $u = 0$; at C, $q = 0$; at A, $q = 1$. We thus get the line CA shown in the ζ plane. From A to J in the z plane we have $q = 1$, whilst $\tan^{-1} v/u$ varies from 90° to zero. Hence in the ζ plane we get the quadrant AJ. We notice that I and J coincide in the ζ plane because the components of velocity are the same at I and J in the z plane. We also see that C is indeterminate in the ζ plane, being, in fact, a quadrant on the circle at infinity, because in the z plane there is no motion at C, and thus there is no definite direction of motion.

But ζ is some function of w, *i.e.* of $\phi + \iota\psi$. Further, the boundaries $ICAJ$ in the z plane are a single stream line along which ψ is a constant which we can choose to be zero. Hence we must obtain such a relation between ζ and w that, if we substitute in w the values of ζ along the curve in the ζ plane, the result shall be a real quantity.

168. **The w Plane.**—The w function can be represented on a w plane in which ϕ and ψ are the real and imaginary co-ordinates. We may thus consider the w plane to represent a case of motion in which the stream lines are all parallel straight lines $\psi =$ constant. The line $\psi = 0$, *i.e.* the ϕ axis, is to correspond to the stream line $ICAJ$ in the z plane. Thus the w plane can be represented physically as the motion obtained when the rigid boundaries IC, CA become one straight line, and AJ is the continuation of this line. To make this more immediately clear we shall letter the parts of the ϕ axis in the w plane as shown in Fig. 88 (c).

Hence we require as follows:

$\zeta =$ a real number, 1 to ∞, must give w negative from $-\infty$ to 0;

$\zeta =$ an imaginary number, $\infty\iota$ to ι, must give w positive from 0 to some quantity $+K$, which depends on the size of the plate AB;

$\zeta = \cos\theta + \iota\sin\theta$, $\theta = \pi/2$ to 0, must give w positive from $+K$ to $+\infty$.

It is readily seen that if we put

$$\frac{K}{w} = \left(\frac{\zeta - \frac{1}{\zeta}}{2\iota}\right)^2,$$

all the requirements are fulfilled. This is frankly a guess. We shall see

later how it can be derived directly ; at present we want to illustrate the general process of the argument.

169. **Normal Stream on a Plate of Finite Width.**—Returning to the problem in Fig. 87, we get the z, ζ, and w planes as shown in

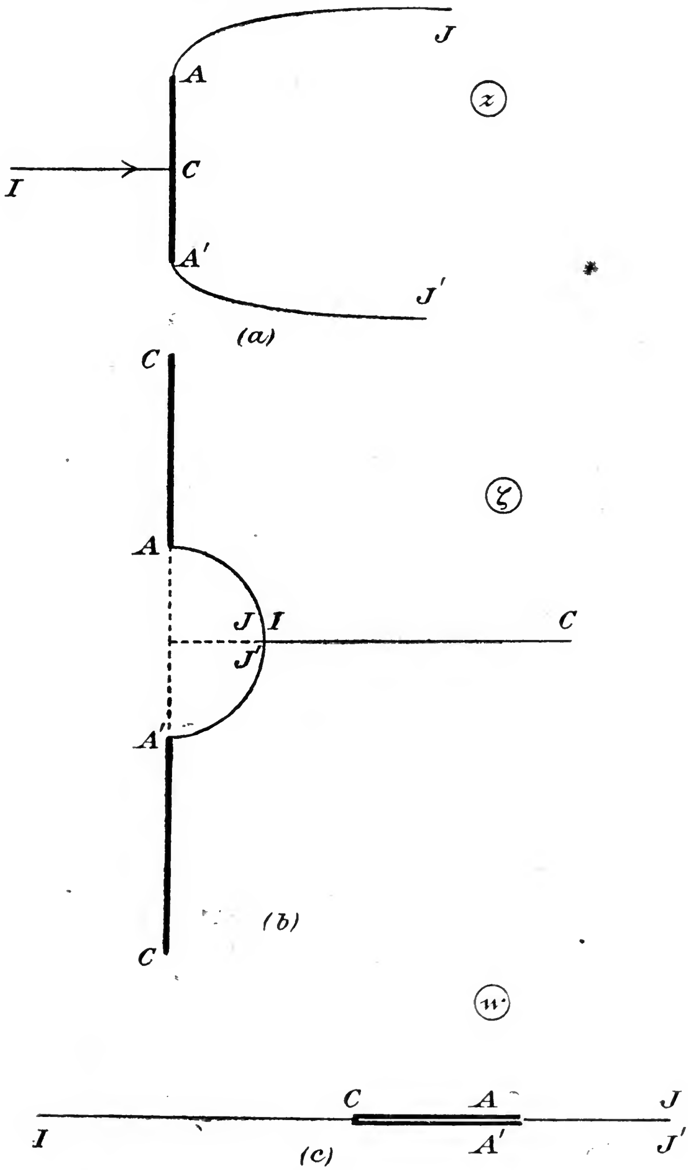

FIG. 89.—z, ζ, and w Planes for the Problem of Fig. 87.

Fig. 89 (a), (b), (c). The w plane can be taken to represent the problem obtained by supposing the plane ACA' in (a) to be hinged at C and to close up so that CA turns through 90° to the right, CA' 90° to the left,

A and A' coming to coincidence: the stream lines AJ, $A'J'$ become coincident straight lines. For this problem we have

$$\frac{K}{w} = \left(\frac{\zeta - \frac{1}{\zeta}}{2\iota}\right)^2 \quad \ldots \ldots \ldots \ldots \quad (40)$$

so that

$$\zeta^2 - 2\iota\sqrt{\frac{K}{w}} \cdot \zeta - 1 = 0,$$

whence

$$\frac{dz}{dw} = \zeta = \frac{\iota\sqrt{K} + \sqrt{w - K}}{\sqrt{w}}, \quad \ldots \ldots \ldots \ldots \quad (41)$$

the positive sign being used, since $w = +\infty$ gives $\zeta = +1$. Hence, using $w = Kw'$,

$$\frac{dz}{dw'} = K\frac{\iota + \sqrt{w' - 1}}{\sqrt{w'}},$$

and z can be obtained in terms of w' by means of an integration; in fact,

$$\frac{z}{K} = 2\iota\sqrt{w'} + \sqrt{w'(w' - 1)} - \cosh^{-1}\sqrt{w'} + \text{a constant}, \quad \ldots \quad (42)$$

the arbitrary constant being of no importance in practice. We can now put $w = \phi + \iota\psi$ and separate z into the real and imaginary parts, thus obtaining x and y as functions of ϕ and ψ. The stream lines ψ=constant and the equipotential lines ϕ = constant can then be plotted.

170. **The Free Stream Line.**—For the purposes of our present work we only need the form of the free stream line and the pressure produced on the plate. The free stream line corresponds to $w = \phi$ (ψ being zero), where ϕ varies from $+K$ to $+\infty$. If, then, ds is an element of length of the free stream line, we have

$$\frac{ds}{d\phi} = \left|\frac{dz}{dw}\right|$$

at the point at which ds is taken, since ds is the modulus of dz, and dw on the free line is $d\phi$. Thus

$$\frac{ds}{d\phi} = |\zeta| = \frac{1}{q} = 1, \quad \ldots \ldots \ldots \quad (43)$$

since the velocity along this stream line is unity. Hence we have

$$s = s_0 + \phi,$$

where s_0 is some arbitrary constant. Also ϕ is given by (40) as

$$\frac{K}{\phi} = \left(\frac{\zeta - \frac{1}{\zeta}}{2\iota}\right)^2,$$

in which for ζ we use the value from the ζ plane, *i.e.* $\zeta = e^{\iota\theta}$. Thus

$$\phi = K\operatorname{cosec}^2\theta, \quad s = s_0 + K\operatorname{cosec}^2\theta \quad \ldots \ldots \ldots \quad (44)$$

This is the intrinsic equation of the free line AJ, s_0 and K being constants, and θ the inclination of the tangent at any point to the axis of x. If we measure s from A, we have $s_0 = -K$ and then $s = K\cot^2\theta$, so that AJ can be plotted as soon as K has been found.

To find K in terms of the width of the plate, we again find $ds/d\phi$. Remembering that along the plate $dz = \iota ds$, we now have by (41)

$$\frac{dz}{dw} = \frac{\iota ds}{d\phi} = \frac{\iota\sqrt{K} + \sqrt{\phi - K}}{\sqrt{\phi}},$$

so that

$$\int^{K} \frac{\sqrt{K} + \sqrt{K - \phi}}{\sqrt{\phi}}\, d\phi$$

is half the width of the plate, say $l/2$. Thus we get, putting $\phi = K\phi'$,

$$l = 2K \int_0^1 \frac{1 + \sqrt{1 - \phi'}}{\sqrt{\phi'}}\, d\phi' = K(\pi + 4),$$

so that

$$K = \frac{l}{\pi + 4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Hence the intrinsic equation of AJ is

$$s = \frac{l}{\pi + 4} \cot^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

171. **The Pressure.**—To find the pressure, we find the difference of the pressures on the two sides of an element ds of the plate. For unit length of a strip ds the difference is $\frac{1}{2}\rho(1 - q^2)\, ds$ (see Chapter V, § 146). Thus the pressure per unit length of the plate is

$$\tfrac{1}{2}\rho \int_{-\frac{l}{2}}^{\frac{l}{2}} (1 - q^2)\, ds = \rho \int_0^{\frac{l}{2}} (1 - q^2)\, ds.$$

But we now have

$$\frac{1}{q} = \frac{ds}{d\phi} = \frac{\sqrt{K} + \sqrt{K - \phi}}{\sqrt{\phi}};$$

hence the pressure is

$$\rho \int_0^K \left(\frac{\sqrt{K} + \sqrt{K - \phi}}{\sqrt{\phi}} - \frac{\sqrt{\phi}}{\sqrt{K} + \sqrt{K - \phi}} \right) d\phi,$$

i.e.

$$2\rho \int_0^K \frac{\sqrt{K - \phi}}{\sqrt{\phi}}\, d\phi = 2\rho K \int_0^1 \sqrt{\frac{1 - \phi'}{\phi'}}\, d\phi'$$

$$= \pi\rho K = \frac{\pi\rho l}{\pi + 4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

We have assumed the velocity at infinity to be unity. If we take the general case, and make this velocity U, the work is exactly similar except that we now make ζ the ratio $\dfrac{dz}{dw}\Big/U$. We find that the free line AJ has the same equation, and that the pressure is now

$$\frac{\pi\rho l U^2}{\pi + 4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

per unit length of the plate. This result is true in any consistent set of

units. Thus, if U is in feet per sec., l in feet, ρ in lb. per cubic foot, the result is in poundals. In a more practical form we can say that the

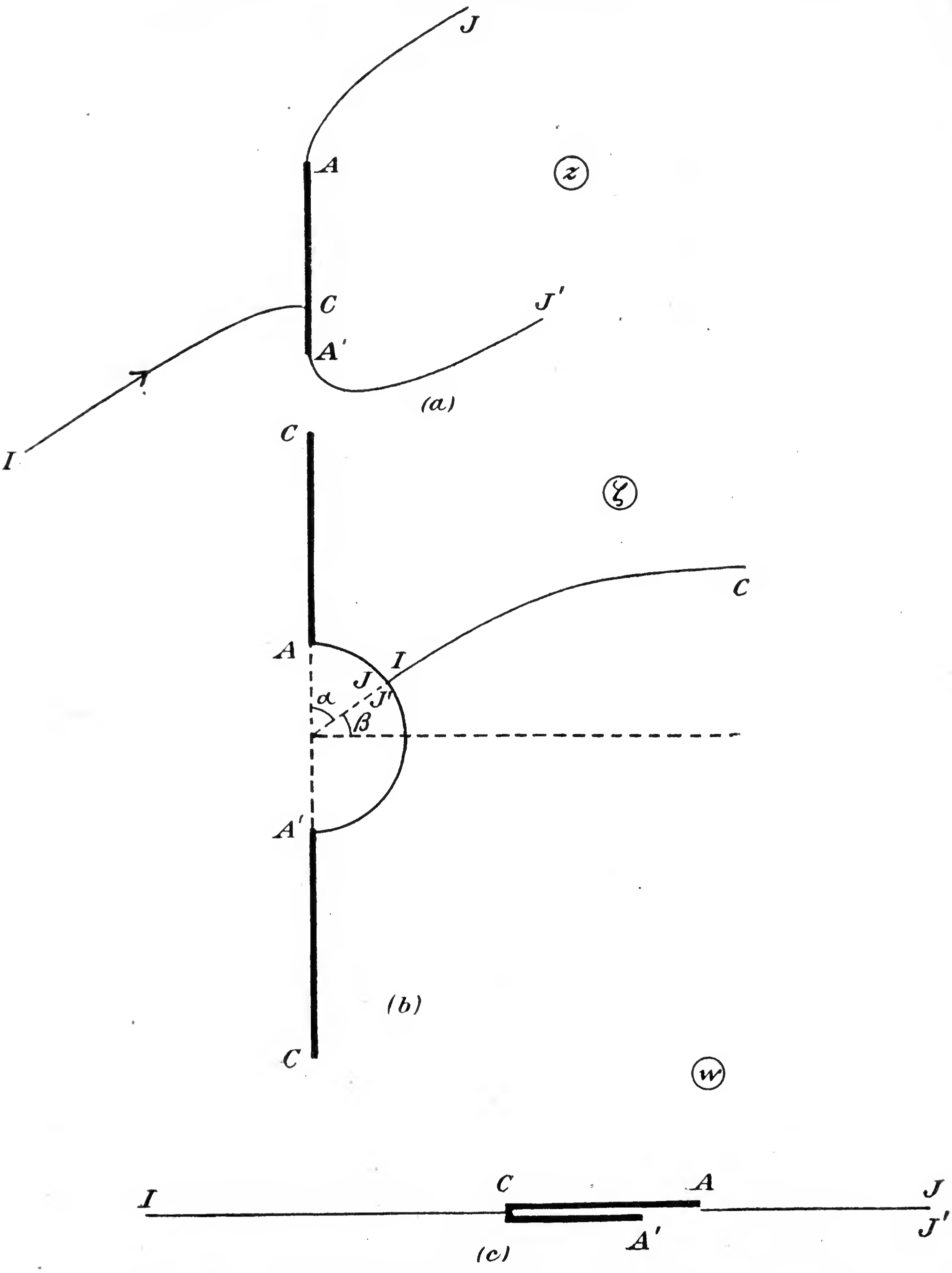

FIG. 90.—z, ζ, and w Planes for Discontinuous Fluid Stream past an Inclined Plane Barrier.

pressure per unit area is $0{\cdot}44\rho U^2/g$ units of weight. In f.p.s. units we get

$$0{\cdot}014\rho U^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

lb. per square foot, U being in feet per sec., and ρ in lb. per cubic foot. The value used in practical aeronautics, based on experiment, is

$$0{\cdot}0194\rho U^2 \text{ (*Aviation Pocket Book*, p. 7)}, \quad \ldots \ldots \quad (50)$$

showing that our result is tolerably correct, considering the large amount of simplification introduced. If the motion is in the plate, the fluid being "at rest," the pressure is the same.

It is important to note that the form of the surface of discontinuity is the same for all velocities and that for all sizes of the plate the dimensions of the surface of discontinuity are proportional to the width of the plate. If we plot it for some chosen value of l, we have it for all values of l on changing the scale of the figure.

172. **Inclined Plate.**—In the more general case where the plate is inclined to the relative wind, the fluid again divides into two parts, but there is no longer any symmetry. Let IC, Fig. 90 (a), be the stream line separating the two parts: at C the velocity is zero. This line then continues along CA and AJ, which is a free line; also along CA' and $A'J'$, the other free line. Taking again unit velocity at infinity, at an angle β with the normal to the plate, so that the "angle of attack" is $\pi/2 - \beta$, the velocity is unity along AJ and $A'J'$, and also ultimately the directions of AJ, $A'J'$ make angles β with the normal. Further, all the stream lines make this angle with the normal where they begin and end, that is, at infinity; this being the boundary condition at an infinite distance.

The z plane is as in Fig. 90 (a). The w plane is obtained by supposing CA and $C'A'$ bent round as in the symmetrical case, the former through $\pi/2 - \beta$, the latter through $\pi/2 + \beta$; but now A and A' will not coincide: the lengths CA, CA' in the w plane (Fig. 90 (c)) will depend on the lengths CA, CA' in the z plane, *i.e.* on the width of the plate and of the angle β.

The ζ plane is shown in Fig. 90 (b). It is similar to the ζ plane in the case of normal incidence (Fig. 89), but the point at which I, J, and J' coincide is now given by $e^{\iota\beta}$, *i.e.* $\cos\beta + \iota\sin\beta$. Since we must now have $w = \infty$ for $\theta = \beta$ on the circle of unit radius in the ζ plane, *i.e.* $\zeta = e^{\iota\beta}$, it is obvious that the form of the relation between w and ζ is

$$\frac{K}{w} = \left(\frac{\zeta - \frac{1}{\zeta}}{2\iota} - \sin\beta\right)^2 . \quad \ldots \ldots \ldots \quad (51)$$

The student should verify that this relation does convert the ζ curve $CAJJ'A'C$ into the positive half of the real axis in the w plane, in the manner indicated. He will find that in the w plane

$$CA = \frac{K}{(1 - \sin\beta)^2}, \quad CA' = \frac{K}{(1 + \sin\beta)^2} \quad \ldots \ldots \quad (52)$$

K depends on the width of the plate. The dividing line IC in the z plane is given by the negative half of the real axis in the w plane. Its form in the ζ plane can be calculated from the equation (51), but it is not needed here.

The free line AJ is, as in § 170, given by

$$s = s_0 + \phi,$$

where

$$\frac{K}{\phi} = (\sin\theta - \sin\beta)^2,$$

θ varying from $\pi/2$ to β. Hence the intrinsic equation of AJ in the z plane is

$$s = s_0 + \frac{K}{(\sin\theta - \sin\beta)^2}.$$

If it is measured from A, we have

$$s = \frac{K}{(\sin\theta - \sin\beta)^2} - \frac{K}{(1-\sin\beta)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

from $\theta = \pi/2$ to β; similarly, $A'J'$ measured from A' is given by the intrinsic equation

$$s = \frac{K}{(\sin\theta + \sin\beta)^2} - \frac{K}{(1+\sin\beta)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

from $\theta = -\pi/2$ to β.

To find K, we have

$$\frac{ds}{d\phi} = \frac{1}{\iota}\frac{dz}{dw} = \frac{\zeta}{\iota},$$

where

$$\zeta^2 - 2\iota\left(\sqrt{\frac{K}{\phi}} + \sin\beta\right)\zeta - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Hence CA in the z plane

$$= \int_0^{\frac{K}{(1-\sin\beta)^2}} \left\{\left(\frac{K}{\phi}\right)^{\frac{1}{2}} + \sin\beta + \sqrt{\left\{\left(\frac{K}{\phi}\right)^{\frac{1}{2}} + \sin\beta\right\}^2 - 1}\right\} d\phi.$$

Using the transformation suggested by the quantity under the square root when worked out, viz.

$$\cos\beta \,.\, \phi^{\frac{1}{2}} = K^{\frac{1}{2}}(\tan\beta + \sec\beta\sin\chi) = K^{\frac{1}{2}}\sec\beta(\sin\beta + \sin\chi),$$

we get

$$CA = K\sec^4\beta\int_{-\beta}^{\frac{\pi}{2}} \{2 + 2\cos(\chi - \beta)\}\cos\chi \,.\, d\chi$$

$$= K\frac{2 + \left(\frac{\pi}{2} + \beta\right)\cos\beta + 4\sin\beta - 2\sin^3\beta}{\cos^4\beta}.$$

Similarly, since CA' is obtained by putting $-\beta$ instead of β, we have CA' in the z plane

$$= K\frac{2 + \left(\frac{\pi}{2} - \beta\right)\cos\beta - 4\sin\beta + 2\sin^3\beta}{\cos^4\beta}.$$

Hence, $l = CA + CA'$ numerically

$$= K\frac{4 + \pi\cos\beta}{\cos^4\beta},$$

i.e.

$$K = \frac{l\cos^4\beta}{4 + \pi\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

For the pressure on the part CA per unit length we have

$$\tfrac{1}{2}\rho\int_0^{CA}(1 - q^2)\,ds,$$

where $1/q = ds/d\phi =$ the value given by (55). Using ϕ as independent variable, the pressure on this part is

$$\tfrac{1}{2}\rho\int_0^{\frac{K}{(1-\sin\beta)^2}}\left(\frac{1}{q}-q\right)d\phi,$$

i.e.

$$\rho\int_0^{\frac{K}{(1-\sin\beta)^2}}\sqrt{\left\{\left(\frac{K}{\phi}\right)^{\frac{1}{2}}+\sin\beta\right\}^2-1}\,d\phi.$$

With the same transformation this integral is easily evaluated and found to be

$$\rho K\sec^3\beta\left(\frac{\pi}{2}-\beta+\sin\beta\cos\beta\right).$$

Similarly, on CA' the pressure is

$$\rho K\sec^3\beta\left(\frac{\pi}{2}+\beta-\sin\beta\cos\beta\right).$$

The total pressure is, therefore, $\pi\rho K\sec^3\beta$, *i.e.*

$$\frac{\pi\rho l\cos\beta}{4+\pi\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

173. **Comparison with Experiment.**—It is usual to measure in terms of the angle of attack. Call this $a(=\pi/2-\beta)$, then we have

$$K=\frac{l\sin^4 a}{4+\pi\sin a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and the pressure per unit length of the plate is

$$\frac{\pi\rho l\sin a}{4+\pi\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If the velocity at infinity is U instead of unity, K is the same, and the pressure is

$$\frac{\pi\rho lU^2\sin a}{4+\pi\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Here, too, the free stream lines are independent of the velocity, and if they are drawn for any assumed width of the plate, they can be used for any other width by a change in the scale.

Calling the pressure for $a=\pi/2$, P_{90°, and the pressure for a, P_a, we have

$$\frac{P_a}{P_{90^\circ}}=\frac{\sin a\,(4+\pi)}{4+\pi\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61$$

To compare this with experiment, we must remember to use the observations for large aspect ratio (the analysis here is for the extreme case of infinite aspect ratio). The best way is to plot P_a/P_{90° as found here, and as found by experimenters. The comparison is given in Fig. 91.

To complete this discussion, we should find also the centre of pressure, *i.e.* the point at which the resultant pressure may be taken to act. The

analysis is heavy :· after some integration, we find that the distance of the centre of pressure from the centre line of the plate is

$$\frac{3}{4}\frac{\cos\alpha}{4+\pi\sin\alpha}l \quad . . . . \quad (62)$$

This is plotted in Fig. 91, and also compared with experiment.

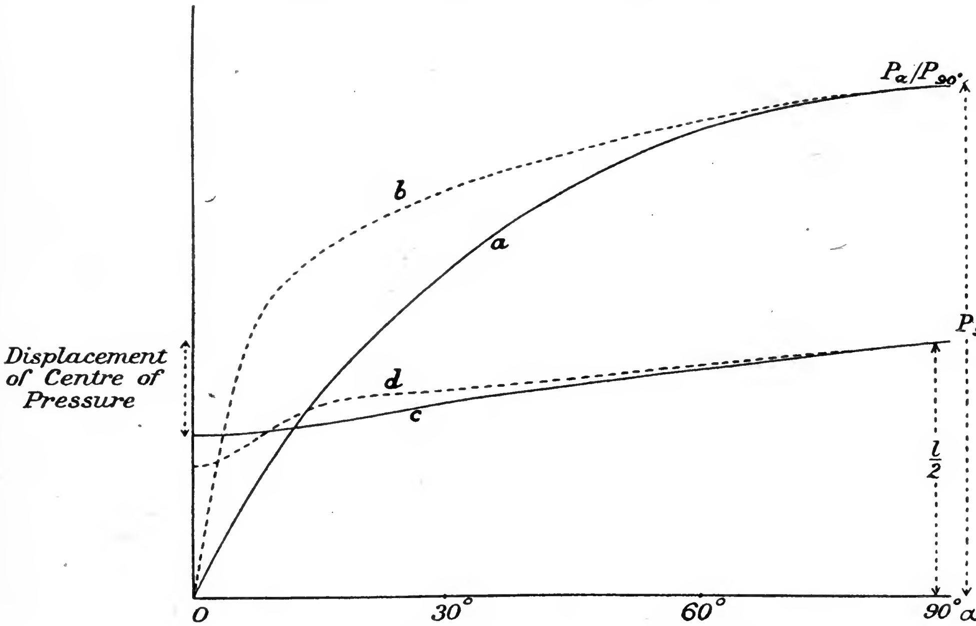

FIG. 91.—Plane Barrier: Theory and Experiment.

(*a*) $P_\alpha P_{90°}$, Theory; (*b*) Experiment, Large Aspect Ratio; (*c*) Displacement of Centre of Pressure, Theory; (*d*) Experiment.

174. **Extension to Curved Barrier.**—If the barrier is not a plane, the mathematical formulation is much more difficult. But the form

$$\frac{K}{w}=\left(\frac{\zeta-\frac{1}{\zeta}}{2\iota}\right)^2$$

at once suggests extensions. The most obvious one is

$$\frac{K}{w}=\left(\frac{\zeta^n-\frac{1}{\zeta^n}}{2\iota}\right)^2, \quad . . . . \quad (63)$$

where n is some number. It is clear that $\zeta=e^{\iota\theta}$ gives $K/w=\sin^2 n\theta$, so that w goes from $+\infty$ to K, J to A in the w plane, Fig. 92, as θ goes from 0 to $\pi/2n$, *i.e.* J to A in the ζ plane. $\zeta=re^{\iota\pi/2n}$, $r=1$ to ∞ gives

$$\frac{K}{w}=\left(\frac{r^n+\frac{1}{r^n}}{2}\right)^2,$$

so that w goes from K to 0, *i.e.* A to C, as ζ goes from A to C in the ζ plane. Similarly, $J'A'$, $A'C$ in the ζ plane are changed into $J'A'$, $A'C$ in the w plane. The line IC is seen to give w negative and with the end

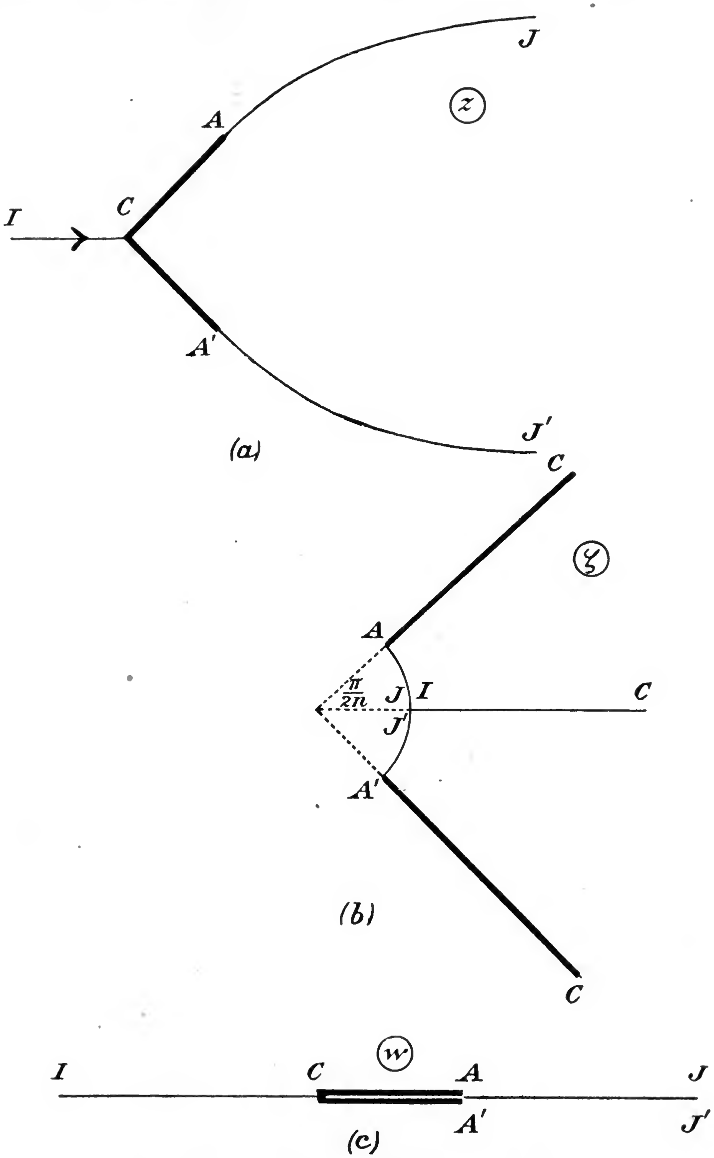

FIG. 92.—Extension of Problem in Fig. 89.

values for IC in the w plane. We can now construct the z plane, and we see that the barrier is in the form of two equal walls forming an angle π/n, the fluid motion being symmetrical, so that IC is the stream line which separates into the free lines AJ, $A'J'$.

This holds so long as $n \nless \frac{1}{2}$. If $n > 1$, we get a barrier convex to the fluid; if $1 > n > \frac{1}{2}$, the barrier is concave to the fluid, as in Fig. 93.

But we note that this generalisation only gives us a barrier formed of two plates at a sharp angle, the fluid moving symmetrically. We need a more generalised method, so as to include non-symmetrical flow and also curved barriers: in practical aeronautics curved or "cambered" wings are universally employed in order to increase the lift of machines.

It is here that we meet one of the limitations of our method. A polygonal boundary can, within certain limits, be discussed. (See, *e.g.*, Greenhill's *Report to Advisory Committee for Aeronautics*—Rep. 19, 1910.) But curved barriers have so far not yielded very useful results. Some considerable progress on general lines has been made by Leathem and others, whilst a helpful process has been devised by H. Levy. (See also a further account by Greenhill.)

175. **Curved Barrier.**—It may be mentioned, however, that the form of the method for plane barriers as presented in this work suggests

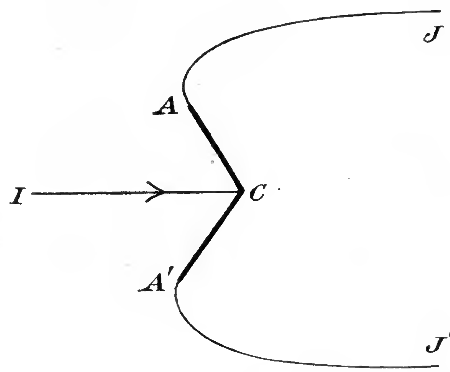

FIG. 93.—Further Extension of Problem in Fig. 89.

an obvious generalisation which can be made to yield results even if *a priori* searching for solutions with given barriers is not provided for. Consider the transformation

$$\frac{K}{w} = \left\{ \frac{F(\zeta) - F\left(\frac{1}{\zeta}\right)}{2\iota} \right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

If $F(\zeta)$ is supposed expanded in powers of ζ, we need only include positive powers, as any negative powers are automatically supplied by $F(1/\zeta)$. Thus, if we had to use $F(\zeta) \equiv a\zeta^m + b/\zeta^n$, we should get

$$F(\zeta) - F\left(\frac{1}{\zeta}\right) = a\zeta^m + \frac{b}{\zeta^n} - \frac{a}{\zeta^m} - b\zeta^n,$$

i.e. $= F_1(\zeta) - F_1(1/\zeta)$, where $F_1(\zeta) \equiv a\zeta^m - b\zeta^n$. We may then write

$$\frac{K}{w} = \left\{ \Sigma \frac{a_n\left(\zeta^n - \frac{1}{\zeta^n}\right)}{2\iota} \right\}^2$$

$$= \left\{ \Sigma\, a_n \sin n\theta \frac{r^n + \frac{1}{r^n}}{2} - \iota \Sigma\, a_n \cos n\theta \frac{r^n - \frac{1}{r^n}}{2} \right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

It has been seen, § 167, that the ζ curve must be such that the corresponding w boundary is the positive half of the real axis in the w plane. To get w real and positive we must make K/w real, *i.e.*

$$\Sigma a_n \cos n\theta \,.\, \frac{r^n - \dfrac{1}{r^n}}{2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

$r = 1$ is one solution. Taking out the factor $r - 1/r$, we have

$$\Sigma a_n \cos n\theta \,.\, \frac{r^{n-1} + r^{n-3} + \ldots + \dfrac{1}{r^{n-3}} + \dfrac{1}{r^{n-1}}}{2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If, then, we start at J in the ζ plane, Fig. 94 (b), and proceed along the circle $r = 1$ with θ increasing from zero, we go along the positive part of

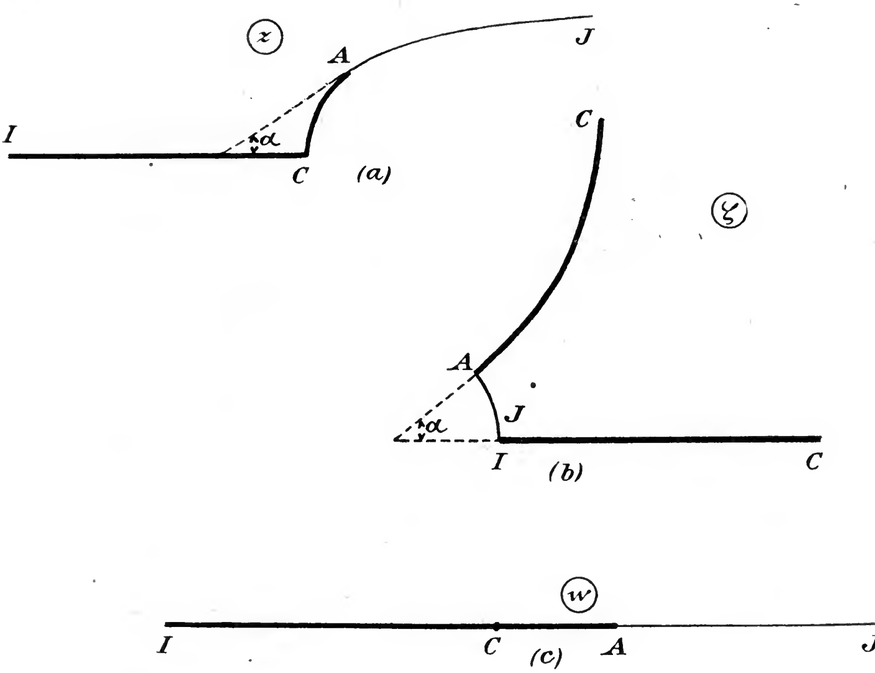

FIG. 94.—Discontinuous Motion Past a Curved Barrier, Symmetrical Case.

the real axis in the w plane, Fig. 94 (c); w commences at ∞ and diminishes, assuming that $\Sigma a_n \sin n\theta$ increases steadily, *i.e.* $\Sigma na_n \cos n\theta$ does not vanish. Let us suppose that at A we get $\Sigma na_n \cos n\theta = 0$, this being the *first* value thus reached which satisfies this equation. But the locus along which, in the ζ plane, we have (66) satisfied must for $r = 1$ give $\Sigma na_n \cos n\theta = 0$. Thus, at A there starts off a curve for which w is real and positive. It is at once seen that this curve starts off radially. If it goes off to infinity with θ increasing steadily as in AC in the ζ plane, then in the w plane we

shall, for suitably chosen values of the coefficients a_n and the exponents, get a steady progression from A to the origin. Further, $\theta = 0$ gives

$$\frac{K}{w} = -\left(\Sigma a_n \frac{r^n - \frac{1}{r^n}}{2}\right)^2,$$

and as r increases from 1 to ∞, w will go from $-\infty$ to 0. We thus get the negative half of the real axis in the w plane. We now see that the

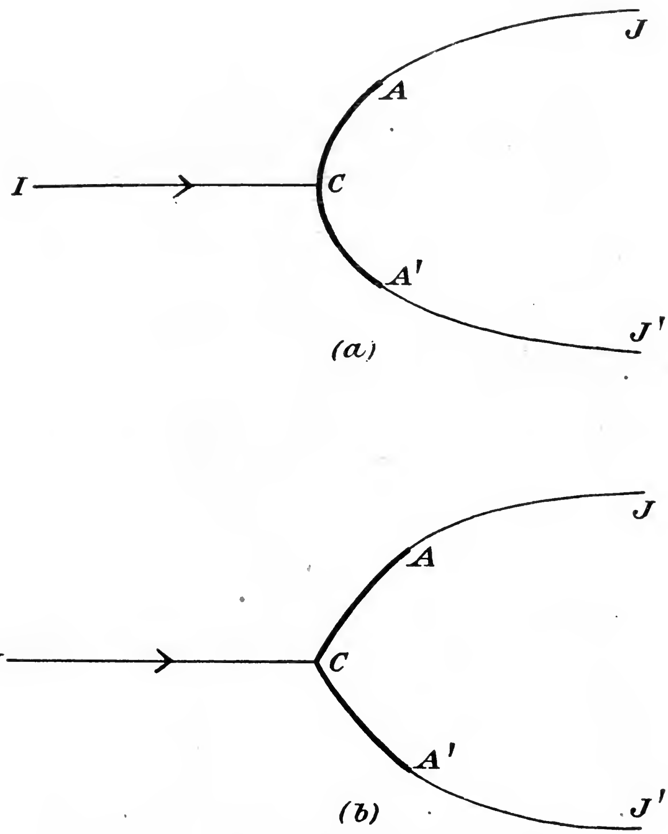

FIG. 95.—Curved Barrier: Symmetrical Case.

corresponding z plane is as in Fig. 94 (a), in which we have a fixed wall IC in the form of a flat plate, a curved wall CA, and a free line AJ, all forming one stream line. By symmetry, we get the case of a wind flowing symmetrically, with unit velocity at ∞, past a curved barrier, which is symmetrical about its middle point (Fig. 95 (a)). The direction at A is the angle AOJ in the ζ plane; the direction at C is parallel to the direction OC in the ζ plane, *i.e.* to the asymptote to the AC curve in the ζ-plane. This may, or may not, be perpendicular to the line of symmetry. The first is shown in Fig. 95 (a), already mentioned: the more general case is shown in Fig. 95 (b).

176. For given values of n and corresponding a_n we can find the free stream lines and the form of the barrier by the same method as in § 170. The free stream line AJ has the intrinsic equation

$$s = s_0 + \frac{K}{(\Sigma a_n \sin n\theta)^2},$$

where K is the constant used in (64), and depends on the *dimension* (as distinct from the shape) of the barrier; s_0 depends on the point from which s is measured. If the angle at a is α, we have for AJ

$$s = \frac{K}{(\Sigma a_n \sin n\theta)^2} - \frac{K}{(\Sigma a_n \sin n\alpha)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

The radius of curvature at any point θ on AJ is

$$-\frac{ds}{d\theta} = 2K \frac{\Sigma n a_n \cos n\theta}{(\Sigma a_n \sin n\theta)^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

which is zero at A and ∞ at J. $J'A'$ is obtained by symmetry when once IC is fixed relatively to AJ.

The barrier can be found because it is a stream line corresponding to a certain range of ϕ, with ψ zero. Hence $ds/d\phi = 1/q$, where q is the resultant velocity, *i.e.* $ds/d\phi = r$, the radius vector in the ζ Argand diagram. Also we have for ϕ the equation

$$\frac{K}{\phi} = \left(\Sigma a_n \sin n\theta \frac{r^n + \frac{1}{r^n}}{2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Hence

$$s = s_1 + \int r\, d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where s_1 is some arbitrary constant depending on where we commence to measure s, and r, ϕ, θ are connected by the equation (70), and the equation

$$\Sigma a_n \cos n\theta \frac{r^n - \frac{1}{r^n}}{2} = 0, \quad r \neq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

We can simplify the general analysis somewhat by putting $r = e^p$. We have

$$s = s_1 + \int e^p d\phi = s_1 + \int (\sinh p + \cosh p)\, d\phi,$$

where

$$\frac{K}{\phi} = (\Sigma a_n \sin n\theta \cosh np)^2, \quad . \quad . \quad . \quad (73)$$

and

$$\Sigma a_n \cos n\theta \sinh np = 0, \quad p \neq 0.$$

The pressure due to the motion is a force parallel to the axis of symmetry equal to

$$\rho \int (1 - q^2) \sin\theta\, ds,$$

integrated over half the barrier, *i.e.* since $q = 1/r = e^{-p}$, $ds/d\phi = r = e^p$, the pressure is

$$2\rho \int \sin\theta \sinh p \,.\, d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

ϕ varying from zero to the value at A.

177. **Example.**—For instance, let us put $F(\zeta) \equiv (\zeta^{\frac{1}{2}} - \zeta)$. Then the AC curve in the ζ plane has the equation

$$\cos\frac{\theta}{2}\frac{r^{\frac{1}{2}} - \dfrac{1}{r^{\frac{1}{2}}}}{2} - \cos\theta\frac{r - \dfrac{1}{r}}{2} = 0, \quad r \neq 1,$$

i.e.

$$\frac{r^{\frac{1}{2}} + \dfrac{1}{r^{\frac{1}{2}}}}{2} = \frac{\cos\dfrac{\theta}{2}}{2\cos\theta}, \qquad \qquad (75)$$

so that the point A is given by the angle a, where $\cos a/2 = 2\cos a$; a is about 65°. The free stream line has the intrinsic equation

$$s = s_0 + \frac{K}{\left(\sin\dfrac{\theta}{2} - \sin\theta\right)^2}, \ \theta \text{ from } 65^\circ \text{ to } 0^\circ \qquad \qquad (76)$$

For the rigid boundary we have to use

$$\frac{K}{\phi} = \left(\sin\frac{\theta}{2}\cosh\frac{p}{2} - \sin\theta\cosh p\right)^2,$$

and by (75)

$$\cosh\frac{p}{2} = \frac{\cos\dfrac{\theta}{2}}{2\cos\theta} \qquad \qquad (77)$$

We get

$$\frac{K}{\phi} = \sin^2\theta\left(1 - \frac{1}{4\cos^2\theta}\right)^2 \qquad \qquad (78)$$

We can now obtain the intrinsic equation of the rigid boundary by substituting from (77) and (78) in the first equation of (73). Its size is involved because of the presence of the constant K.

Also, if we substitute for p, ϕ in terms of θ, in the expression (74), we get the value of the pressure as a single quadrature. Numerical or graphical integration may be resorted to, both for the pressure and for plotting the boundaries.

178. In a similar way, more difficult cases can be discussed. The method can also be used for oblique incidence, if we extend equation (64) into the form

$$\frac{K}{w} = \left\{\frac{F(\zeta) - F\left(\dfrac{1}{\zeta}\right)}{2\iota} - \frac{F(e^{\iota\beta}) - F(e^{-\iota\beta})}{2\iota}\right\}, \qquad \qquad (79)$$

where β is an angle like that defined in § 172.

179. **Biplanes, etc.**—As far as a monoplane wing is concerned, the above analysis is sufficient. When we come to consider biplane and triplane wings, we must either assume that the superposed wings have no effect on one another's pressure, which is certainly untrue, both *a priori* by theory, and also as proved by experiment, or we must invent a process for taking into account the gap between the wings. This at once suggests that we should start with the escape of fluid through an opening in a barrier.

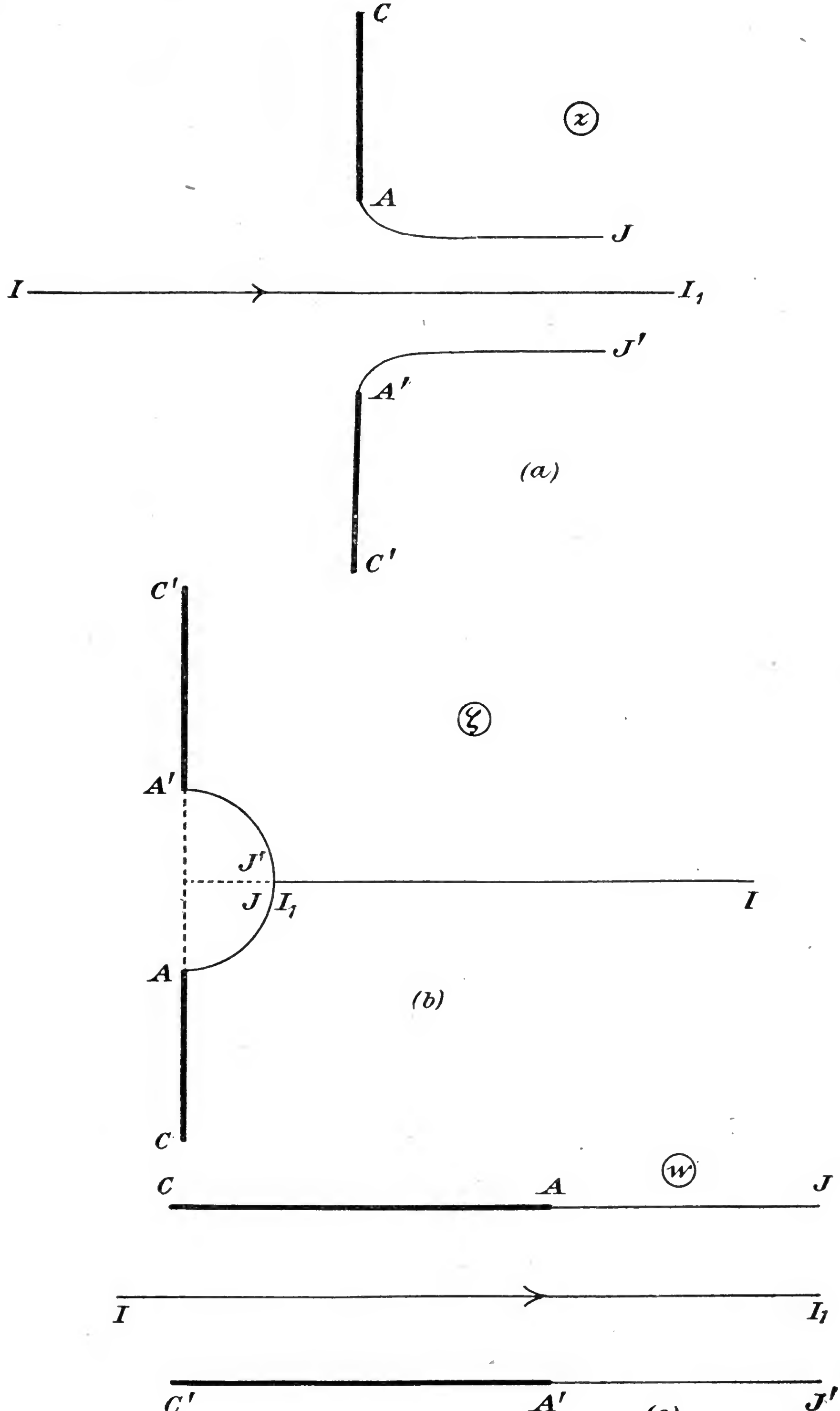

Fig. 96.—Discontinuous Motion through a Gap in a Plane Wall.

The problem we shall now discuss is the one obtained as a sort of correction of the work in Chapter V., §§ 161–63. Let the barrier be represented by the y axis in the z plane, Fig. 96 (a), the opening being obtained by the removal of the part AA'. As the barrier is of infinite extent in both directions AC, $A'C'$, the motion must be symmetrical about a straight stream line II_1. Let AJ, $A'J'$ be the free stream lines. We take the velocity at the infinite end of the jet to be unity, so that it is also unity along the free lines AJ, $A'J'$. The velocity at infinity on the left of the barrier is zero.

The corresponding w plane, Fig. 96 (c), is obtained by supposing CA, $C'A'$ to be bent round to the left till they are parallel, the free lines AJ, $A'J'$ being now continuations of CA, $C'A'$. The ζ plane is at once seen to be as shown in Fig. 96 (b). In this problem the ζ curve, which gives the boundary conditions, viz. $CAJJ'A'C'$, must give *two* parallel lines in the w plane, since the free lines in the z plane are not the same stream line, but distinct. If we take the w plane to be symmetrically arranged about the real (ϕ) axis, we must have for CAJ the form $\phi + \iota K\,\pi/2$, and for $C'A'J'$ the form $\phi - \iota K\,\pi/2$, where ϕ is real, and varies from $-\infty$ to $+\infty$, and K is a constant such that $AA' = K\pi$.

The only function we are familiar with that gives us a real varying quantity + a fixed imaginary part is the logarithm. We, therefore, use such a logarithm as $\log_e(\zeta - 1/\zeta)$. After a little experimenting we find that

$$w = -K\log_e\left(\frac{\zeta - \dfrac{1}{\zeta}}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

is the transformation we need. Thus for $C'A'$ in the ζ plane, $\zeta = \iota r$, $r = \infty$ to 1, so that we have

$$w = -K\log_e\left(\frac{r + \dfrac{1}{r}}{2}\iota\right) = -K\left(\log_e\frac{r+\dfrac{1}{r}}{2} + \frac{\iota\pi}{2}\right),$$

the real part varying from $-\infty$ to 0; we get $C'A'$ in the w plane. For $A'J'$ in the ζ plane we have $\zeta = e^{\iota\theta}$, $\theta = \pi/2$ to 0. w is

$$-K\log_e(\iota\sin\theta) = K\left(\log_e\operatorname{cosec}\theta - \frac{\iota\pi}{2}\right),$$

the real part varying from 0 to $+\infty$; we get $A'J'$ in the w plane. In the same way it is seen that CA, AJ in the ζ plane become CA, AJ in the w plane. We have then ζ in terms of w for the z plane, and we get z in terms of w; the mathematical formulation of the problem is complete.

The interesting thing in this case is to get the form of the free lines. We choose $A'J'$ since θ is positive, and $< 90°$ on this line, not on the other. As before, we have for this line $ds/d\phi = 1$, so that $s = s_0 + \phi$. But for ϕ we now have

$$\phi - K\iota\frac{\pi}{2} = -K\log_e(\iota\sin\theta),$$

where θ varies from 90° to 0. Hence we have $\phi = K \log_e \operatorname{cosec} \theta$, and the intrinsic equation of $A'J'$ is

$$s = s_0 + K \log_e \operatorname{cosec} \theta.$$

If we measure s from A', where $\theta = 90°$, we get $s_0 = 0$, and

$$s = K \log_e \operatorname{cosec} \theta \quad \text{. . . . . (81)}$$

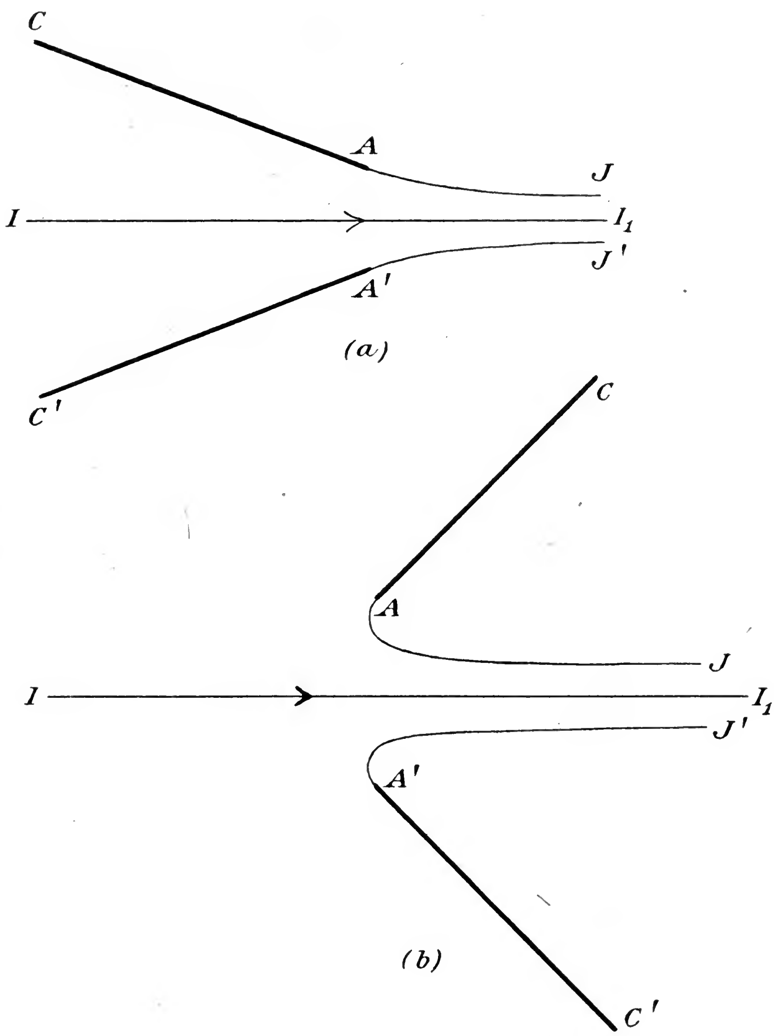

FIG. 97.—Extension of Problem of Fig. 96.

It can be shown by methods similar to those in §§ 170, 171 that the ultimate width of the jet at infinity is 0·611 of the width of the opening AA'. For the purpose of our present subject this would be too great a digression to tolerate.

180. **Extension.**—As in § 174, we obtain generalised problems by substituting some function $F(\zeta)$ for ζ in (80). If we take, *e.g.*,

$$w = -K\log_e\left(\frac{\zeta^n - \dfrac{1}{\zeta^n}}{2}\right), \qquad (82)$$

we get the case of a barrier in the form of two planes inclined to one another with a symmetrical opening, the fluid motion being, of course, still symmetrical. The index n must be $\nless \frac{1}{2}$; when it is > 1, the motion is as in Fig. 97 (a); when it is < 1, $> \frac{1}{2}$, the motion is as in Fig. 97 (b). The student ean easily verify these statements.

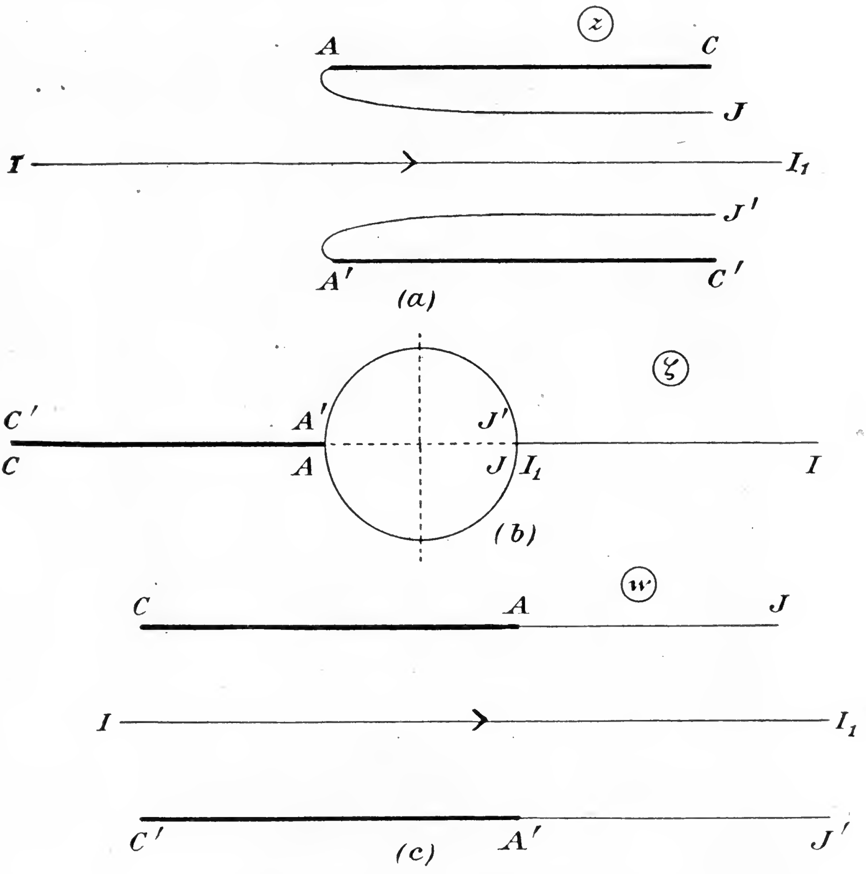

Fig. 98.—Extreme Case of Problem in Fig. 97.

The important case arises when $n = \frac{1}{2}$. The z plane is as in Fig. 98 (a), the motion being comparable with that of water out of a tap which projects far into the vessel containing the water. The ζ and w planes are given in Fig. 98 (b), (c). In this case the ultimate width of the jet is half of the distance between the two parts of the barrier (*Borda's Mouthpiece*). By assuming more generalised forms of $F(\zeta)$ we can discuss curved walls.

181. **Finite Barriers.**—But in a biplane we have two *finite* barriers; we ought then to attempt a mathematical formulation of the problem

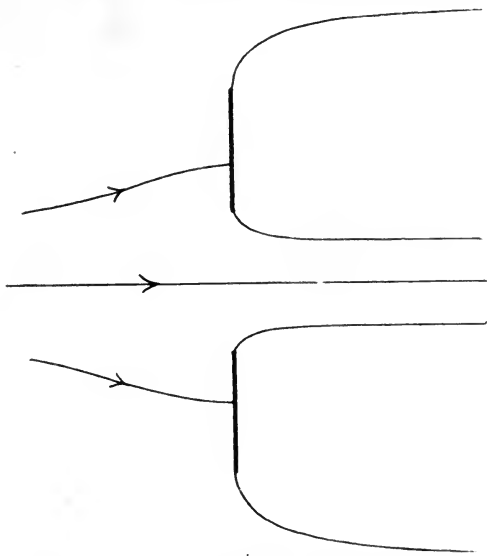

FIG. 99.—Discontinuous Motion Past Two Barriers.

shown in Fig. 99. But elliptic integrals are required for this case, and the student is referred to Greenhill's Report for such and other problems.

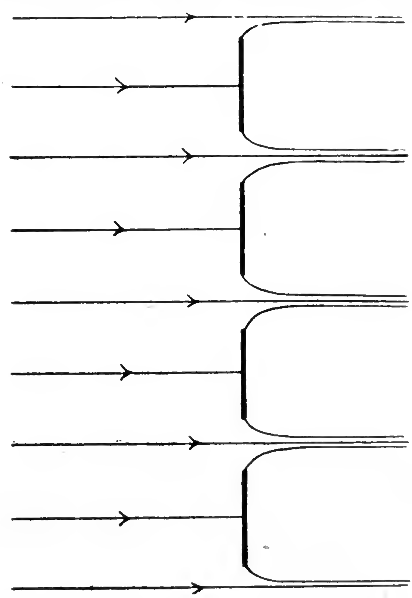

FIG. 100.—Discontinuous Motion Past an Indefinite Number of Equal Barriers, Parts of one Wall, at Equal Intervals.

A simplified case is presented if we imagine a lattice-work consisting of an infinite number of equal plane barriers, all in one plane at equal

distances apart, Fig. 100. It is clear that we get in each interval a straight stream line, and by considerations of symmetry it follows that we only need to discuss the problem as defined in Fig. 101 (a), where IC, CA, I_1J_1 are given rigid walls, IC parallel to I_1J_1, and CA perpendicular. AJ is

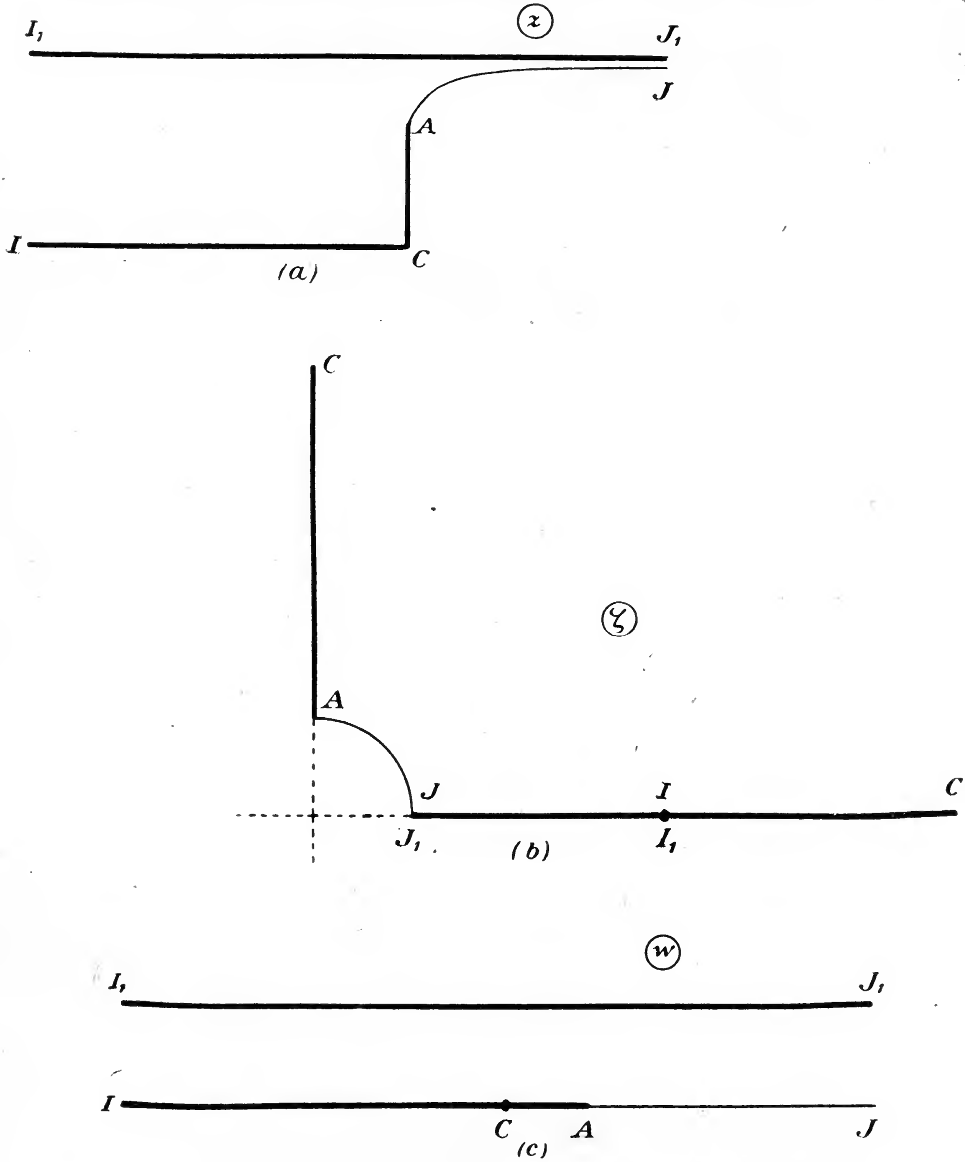

FIG. 101.—z, ζ, and w Planes for Problem of Fig. 100.

the free stream line. For ζ and w planes we have Fig. 101 (b), (c). In the w plane CA is turned till it is in the same straight line as IC, and AJ is the continuation. In the ζ plane there is no difficulty about CA, AJ, it being assumed that the velocity is unity at the free line in the z plane.

The velocity at infinity to the left, viz. I and I_1, is smaller than at infinity to the right, viz. J and J_1. Hence, in the ζ plane I and I_1 are at a point on the continuation of the radius along the positive rea axis; the position of this point depends on the width of the wall CA.

182. The transformation for this case is too difficult to guess. If we write

$$w = K \log_e \left\{ 1 + \frac{k^2}{\left(\dfrac{\zeta - \dfrac{1}{\zeta}}{2\iota}\right)^2} \right\}, \quad \ldots \ldots \ldots \quad (83)$$

where K defines the size, and k is a constant depending on the width of CA compared to the distance between the plane barriers, we find that all the conditions of the problem are satisfied. To simplify, we put $K = 1$, since K merely affects the scale of the diagram. Now CA in the ζ plane has $\zeta = \iota r$, $r = \infty$ to 1; hence

$$w = \log_e \left\{ 1 + \frac{k^2}{\left(\dfrac{r + \dfrac{1}{r}}{2}\right)^2} \right\},$$

and it is real and positive going from 0 to $\log_e(1 + k^2)$, *i.e.* from the origin C to A in the w plane, where $CA = \log_e(1 + k^2)$. AJ in the ζ plane gives $w = \log_e(1 + k^2 \operatorname{cosec}^2 \theta)$, $\theta = \pi/2$ to 0, so that w is real and positive going from $\log_e(1 + k^2)$ to ∞, *i.e.* AJ in the w plane. For the real axis in the ζ plane put $\zeta = r$; then

$$w = \log_e \left\{ 1 - \frac{k^2}{\left(\dfrac{r - \dfrac{1}{r}}{2}\right)^2} \right\}.$$

So long as $(r - 1/r) < 2k$, w is the logarithm of a negative quantity, and we write it

$$w = \log_e \left\{ \frac{k^2}{\left(\dfrac{r - \dfrac{1}{r}}{2}\right)^2} - 1 \right\} + \pi\iota,$$

the real part varying from $+\infty$ at $r = 1$ to $-\infty$ at $r = k + \sqrt{k^2 + 1}$, giving us the line I_1J_1 in the w plane at distance $\pi\iota$ from the real axis. When $(r - 1/r)$ becomes $> 2k$, w is the logarithm of a real quantity which is less than 1, so that w is real and negative, going from $-\infty$ at $r = k + \sqrt{k^2 + 1}$ to 0 at $r = \infty$; we, therefore, get the negative half of the real axis, IC, in the w plane.

The mathematical formulation is complete. It refers, of course, also to the case of a plane barrier placed symmetrically in an infinite canal with straight walls, obtained by duplicating Fig. 101 (a) about IC, and demolishing the wall IC, which is now unnecessary, by symmetry.

The physical meaning of k is given by the fact that $k + \sqrt{k^2 + 1}$ is the ratio of the radius vector of I in the ζ plane to the radius vector of J. Hence the velocities at I, J in the motion are the ratio $\sqrt{k^2 + 1} - k$, and the width I_1I at infinity is $k + \sqrt{k^2 + 1}$ times the width J_1J at infinity. To get a complete notion we must find CA in terms of k.

If, now, we use a more generalised function $F(\zeta)$ instead of ζ in the relation (83), we get the case of a more generalised barrier in the canal. The student will at once interpret the substitution of ζ^n for ζ, where $n > \frac{1}{2}$. For other forms of $F(\zeta)$ we get motions with curved barriers.

183. **Direct Derivation of the w, ζ Relation.**—We cannot leave the student in the dark as to the direct discovery of the relation between ζ and w in cases where this can be done *a priori*. This is especially

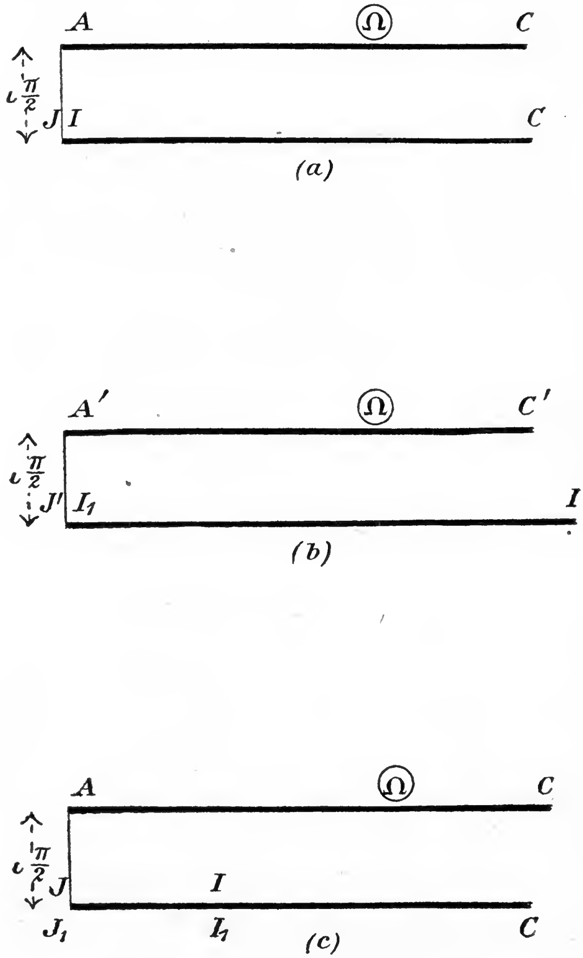

FIG. 102.—Ω Planes for Problems of (*a*) Fig. 88, (*b*) Fig. 90, (*c*) Fig. 101.

applicable to the transformation (83), which can hardly be offered as a *bona-fide* guess. It is, in fact, a modification of a form used by Love. The method now to be given only refers to problems with plane barriers; it will be seen that the method itself imposes its own limitations.

In any problem with plane barriers and free stream lines the ζ curve consists of a part or parts of the circle $r = 1$, and continuations of its

radii. Thus, on any part of the ζ curve we either have r constant or θ constant. In Fig. 89 (*b*), *e.g.*, we have $\theta = \pi/2$ for CA, $r = 1$ for $AJJ'A'$, $\theta = -\pi/2$ for $A'C$; in Fig. 92 (*b*) we have $\theta = \pi/2n$ for CA, $r = 1$ for $AJJ'A'$, $\theta = -\pi/2n$ for $A'C$; and similarly in any other such case. The value of ζ on any one branch or part of a ζ curve is, therefore, either a constant times $e^{\iota\theta}$, or r multiplied by a constant exponential.

184. **The Ω Function.**—Now $\log_e \zeta = \log_e r + \iota\theta$; hence on any one branch of a ζ curve $\log_e \zeta$ is either a constant $+ \iota\theta$, or $\log_e r +$ a constant (usually imaginary or zero). If, then, we draw another Argand diagram for $\log_e \zeta$, which we call Ω, the ζ curve will become an Ω locus consisting of a number of straight lines. Thus, taking the three typical problems we have discussed, the Ω locus for the ζ curve in Fig. 88 (*b*) is as in Fig. 102 (*a*); the Ω locus for half of the ζ curve in Fig. 96 (*b*) is as in Fig. 102 (*b*); for the ζ curve in Fig. 101 (*b*) it is as in Fig. 102 (*c*).

If, then, we find a relation between Ω and w we have the required

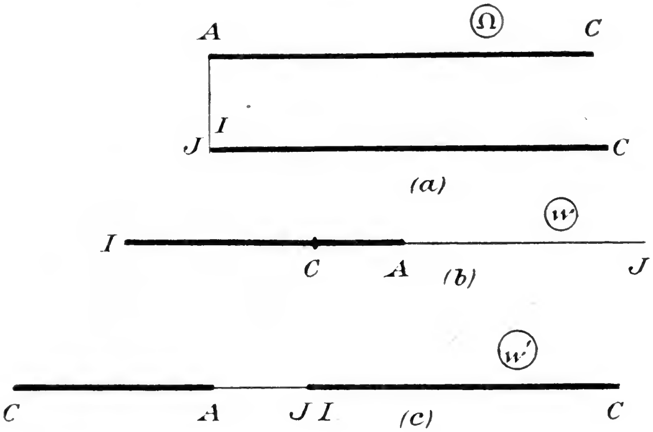

FIG. 103.—Problem of Fig. 88; $w = -\dfrac{1}{w'}$.

The Ω and w' Planes can be directly connected by a Schwartz-Christoffel Transformation.

relation between ζ and w. The boundary locus in the w plane also consists of straight lines. The analytical problem is: to find a relation between Ω and w such that the polygon in one plane becomes the corresponding polygon in the other.

185. **Schwartz-Christoffel Transformation: The *t* Plane.**—There is no way of connecting directly two polygonal boundary loci in general. A method does exist, however, for converting any boundary into the real axis in some Argand diagram, which we shall call the t plane. The t plane can thus be used as an intermediary between the Ω and w planes; find the relation between Ω and a corresponding t plane, and between w and the same t plane; the result gives the relation between Ω and w, and, therefore, between ζ and w.

186. **Plane Barrier of Finite Width.**—A somewhat simplified procedure is possible in the case of the problem of direct incidence on a flat plate. The Ω and w planes are shown in Fig. 103 (*a*), (*b*). Let us take the origin of w to be at C, and let us write $w = -1/w'$. Then we have

the w' plane in Fig. 103 (c), the boundary conditions in the w' plane being expressed by the real axis $CAIJC$. We have, then, to connect by means of some relation between Ω and w' the boundary conditions $CAIJC$ in the Ω plane into the boundary conditions $CAIJC$ in the w' plane.

It will be noticed that in order to get the Ω boundary conditions converted into the w' conditions, it is necessary to straighten out at A and IJ internal angles of 90°. The Schwartz-Christoffel method is based on the fact that such a straightening out is made analytically possible by means of a certain type of differential relation. Put

$$\frac{d\Omega}{dw'} = \lambda (w' - a')^{\alpha} (w' - b')^{\beta}, \quad \text{(84)}$$

where a' is the value of w' at A, b' is the value of w' at IJ, and λ is a constant. Consider the change in the *argument* of $d\Omega/dw'$ as w' travels from C to A, then to IJ, then to C in the w' plane. From C to A, w' is algebraically less than a' and b'. Hence the quantities $w' - a'$, $w' - b'$, retain their signs. As we pass A, $w' - a'$ changes its sign, whilst $w' - b'$ retains its sign; there is no change till we reach IJ; as we pass IJ, $w' - b'$ changes its sign, whilst $w' - a'$ retains its sign. Now, the *argument* of our expression (84) is $\alpha \times$ the *argument* of $(w' - a') + \beta \times$ the *argument* of $(w' - b')$, since the argument of a product is the sum of the arguments of the factors. Hence, as we go from C to A in the w' plane the argument of $d\Omega/dw'$ does not change; at A it changes suddenly by $\alpha \times$ change in the argument of $(w' - a') = \alpha\pi$; from A to IJ the argument does not change further; then at IJ it changes suddenly by $\beta \times$ change in the argument of $(w' - b') = \beta\pi$; then, finally, from IJ to C the argument does not change. Since dw' is always positive, and therefore its argument does not change, this means that $d\Omega$ goes through these changes in argument, and thus the Ω diagram gives straight lines CA, AIJ, IJC. But the internal angles in the Ω figure, *i.e.* the changes in the argument of $d\Omega$, are $-\pi/2$ at A, $-\pi/2$ at IJ (the positive argument being always taken as anticlockwise). Hence we have

$$\alpha = \beta = -\tfrac{1}{2},$$

so that

$$\frac{d\Omega}{dw'} = \frac{\lambda}{\sqrt{(w' - a')(w' - b')}}, \quad \text{(85)}$$

where a', b' are the values of w' at A, IJ respectively. In our case $b' = 0$, since we "inverted" in the w plane, so that IJ go to the origin in the w' plane. Thus we get

$$\frac{d\Omega}{dw'} = \frac{\lambda}{\sqrt{w'(w' - a')}},$$

whence, integrating, we deduce

$$\Omega = \lambda \cosh^{-1}\left(\frac{2w' - a'}{a'}\right) + \log_e \mu,$$

where μ is an arbitrary constant. This gives

$$\log_e\left(\frac{\zeta}{\mu}\right) = \lambda \cosh^{-1}\left(\frac{2a}{w} - 1\right), \quad \text{(86)}$$

where a is the value of w at A. To evaluate the constants λ, μ we use

the facts that $\zeta = \iota$ gives $w = a$ (see Fig. 88), so that $\mu = \iota$, and $\zeta = 1$ gives $w = \infty$, so that $\lambda = \frac{1}{2}$. Hence

$$\frac{2a}{w} = 1 + \cosh\left(2 \log \frac{\zeta}{\iota}\right),$$

so that

$$\frac{a}{w} = \left(\frac{\frac{\zeta}{\iota} + \frac{\iota}{\zeta}}{2}\right)^2 = \left(\frac{\zeta - \frac{1}{\zeta}}{2\iota}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

This is the transformation used above, § 170, with K instead of a.

187. **Gap in a Plane Wall.**—The argument of the last article was possible because we could change the boundary conditions in the w plane into such boundary conditions in the w' plane that the relation

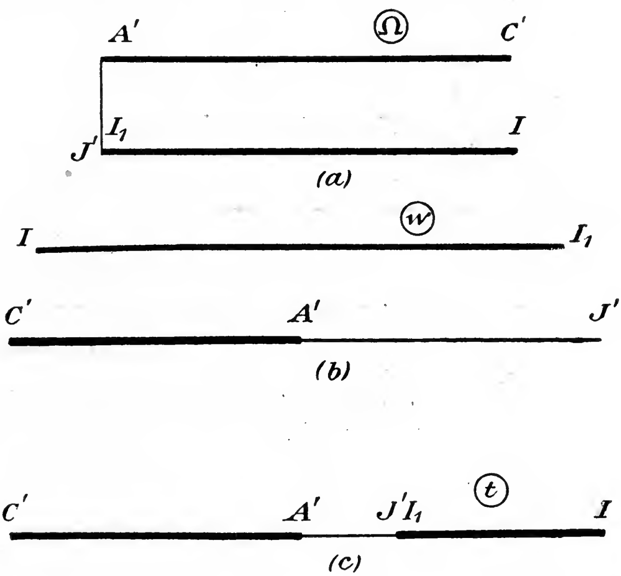

FIG. 104.—Problem of Fig. 96. The Ω Plane can be connected with each of the w and t Planes respectively.

between Ω and w' could be obtained easily and directly. In general, it is necessary to use the Schwartz-Christoffel method both for Ω and for w. We now take the case of fluid escaping through a gap in a plane wall. The Ω and w planes are given in Fig. 104 (a), (b). Since the w boundary is now not the real axis we cannot proceed directly. Let us, then, define a t plane, Fig. 104 (c), in which the boundary conditions are given by the real axis $C'A'J'I_1I$.

By analogy with § 186 we have at once

$$\frac{d\Omega}{dt} = \frac{\lambda}{\sqrt{(t-a)(t-b)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where a, b are the values of t at the points A', $J'I_1$ in the t plane. For the relation between w and t we note that A' in w and A' in t are such that there are no changes in the arguments of dw and dt. Hence A' does not enter into the relation. It is only necessary to make $J'I_1I$ in the

t plane bend round through 180° at $J'I_1$. We therefore need a change $+\pi$ in the argument of dw at t goes through $J'I_1$. This gives

$$\frac{dw}{dt} = \frac{\nu}{t-b}, \qquad (89)$$

where ν is an arbitrary constant, by exactly the same argument as in § 186.

Choose the origin of t at $I'J'$, then $b = 0$. Also choose the scale of t so that A' is the point -1. We get

$$\frac{d\Omega}{dt} = \frac{\lambda}{\sqrt{t(t+1)}}, \quad \frac{dw}{dt} = \frac{\nu}{t},$$

so that

$$\log_e\left(\frac{\zeta}{\mu}\right) = \lambda \cosh^{-1}(2t+1), \quad w = \nu \log_e\left(\frac{t}{\tau}\right), \qquad (90)$$

where μ, τ (and also λ, ν) are arbitrary constants.

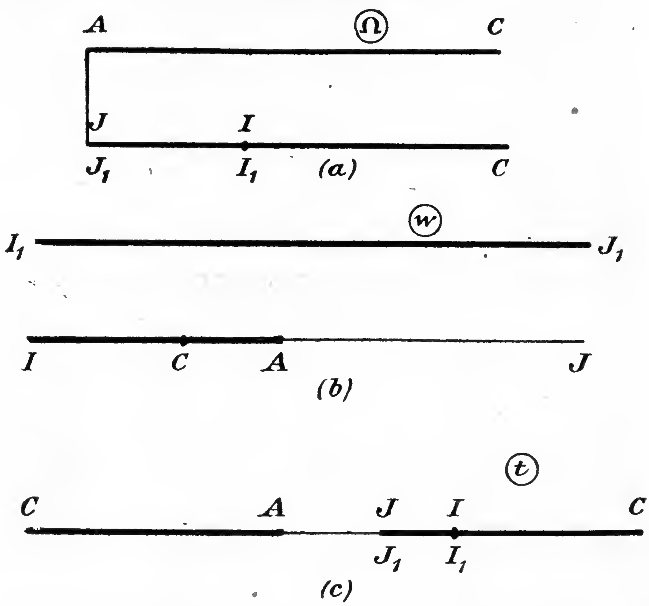

FIG. 105.—Problem of Fig. 101. The Ω Plane can be connected with the w, t Planes respectively.

But if we refer to Fig. 96 we see that $\zeta = 1$ gives $t = 0$, and $\zeta = \iota$ gives $t = -1$; hence we readily find $\mu = 1$, $\lambda = \frac{1}{2}$, so that

$$t = \left(\frac{\zeta - \frac{1}{\zeta}}{2}\right)^2 \qquad (91)$$

Also, we can choose the origin of w as we like, so that τ is at our disposal. Writing $\tau = 1$, and using the notation $\mu = -K$, we have the relation (80) used in § 179.

188. **The "Biplane" Problem.**—The third type of problem we have been led to investigate is given by Fig. 101. The Ω and w planes are given in Fig. 105 (a), (b). Choose a t plane as shown in Fig. 105 (c).

We again have at once

$$\frac{d\Omega}{dt}=\frac{\lambda}{\sqrt{(t-a)(t-b)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

where a, b are the values of t at A and JJ_1 respectively. To get the w boundary conditions from those of t we note that it is necessary to make the argument of dw change by $+\pi$ as t passes through JJ_1 and II_1 respectively. Thus we must write

$$\frac{dw}{dt}=\frac{\nu}{(t-b)(t-c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

Choose the origin of t at JJ_1, so that $b=0$, and let A have the value $t=-1$. Then

$$\frac{d\Omega}{dt}=\frac{\lambda}{\sqrt{t(t+1)}},\quad \frac{dw}{dt}=\frac{\nu}{t(t-c)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

where c is the value of t at II_1. We find that once again

$$\log_e\left(\frac{\zeta}{\mu}\right)=\lambda\cosh^{-1}(2t+1).$$

Referring to Fig. 101, we see that $\zeta=1$ gives $t=0$, $\zeta=\iota$ gives $t=-1$; thus, as in § 187,

$$t=\left(\frac{\zeta-\dfrac{1}{\zeta}}{2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Integrating the relation between w and t, we find

$$w=\frac{\nu}{c}\log_e\left(1-\frac{c}{t}\right)+\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

where ν, τ are really arbitrary. Write $\nu/c=K$, $c=k^2$, and choose the origin of w so that $\nu=0$. We find

$$w=K\log_e\left\{1-\frac{k^2}{\left(\dfrac{\zeta-\dfrac{1}{\zeta}}{2}\right)^2}\right\}=K\log_e\left\{1+\frac{k^2}{\left(\dfrac{\zeta-\dfrac{1}{\zeta}}{2\iota}\right)^2}\right\}, \quad . \quad . \quad . \quad (97)$$

which is the relation used in § 182.

Note.—The fact that C in the w plane corresponds to the two infinite ends of the real axis in the t plane does not enter into the relation between w and t, as, mathematically speaking, the two ends C C are really the same point and there is no discontinuity.

189. **Generalisation.**—We are now in a position to sum up the Schwartz-Christoffel method in a general formula. The Ω and w boundary conditions are certain polygons with known changes of direction at the angular points. Choose the t plane on the assumption that the boundary conditions become the real axis. Then we write

$$\frac{d\Omega}{dt}=\frac{\lambda}{(t-a)^{\alpha/\pi}(t-b)^{\beta/\pi}(t-c)^{\gamma/\pi}\ldots},$$

where λ is an arbitrary constant; $a, b, c\ldots$ are the values of t corresponding to the angular points of the Ω polygon; and $\alpha, \beta, \gamma\ldots$ are the

changes in direction at the angular points of the Ω polygon, it being assumed that we go round it in the conventional positive sense, *i.e.* keeping the internal area on the left. Similarly,

$$\frac{dw}{dt}=\frac{\nu}{(t-a)^{\alpha'/\pi}(t-b)^{\beta'/\pi}(t-c)^{\gamma'/\pi}\ldots},$$

where ν is an arbitrary constant, and α', β', γ' . . . are the corresponding angles in the w polygon. Care must be taken to include all the angular points that occur. Thus in § 188 H_1 is not an angular point in Ω, but it is an angular point in w.

The application of this method to a number of interesting cases is discussed in Greenhill's Report already referred to.

The value of the method is entirely dependent upon the fact that we have *rectilinear* polygons to transform into one another by means of the intermediate t plane. Thus it can only be used where the ζ plane gives boundary conditions in the form of straight lines through the origin and arcs of a circle with centre at the origin. The rigid boundaries in the z plane are given by some, or all, of the radial straight lines in the ζ plane; hence the rigid boundaries must be plane. It is thus proved that we cannot use this method for anything but plane barriers.

Reference has already been made to methods of dealing with curved barriers. The subject is one to which mathematicians have begun to devote their attention, and it is one that offers a fruitful field for useful research.

Rotational Motion and Viscosity.—No appreciable progress can be recorded of the useful practical solution of problems of fluid pressure in which the assumption of irrotational motion in a non-viscous fluid is discarded. We shall, therefore, at once proceed to the application of our results (including experimental facts) to the problems of aeroplane motion. For the theory of rotational motion and the theory of viscosity, the student is referred to standard books like Lamb's *Hydrodynamics.*

SECTION III

AEROPLANE MOTION

CHAPTER VII

STEADY MOTION AND STABILITY: THE PROPELLER

190. **The "a priori" Study of Aeroplanes.**—Having examined the theoretical basis of the dynamics of aeroplanes, we shall proceed to a discussion of the application of experimental methods and results to the more detailed study of actual aeroplane problems.

We have seen, § 173, that aerodynamical theory suggests (at least as an approximation) that the air pressure on a flat plate whose angle of attack is a, the motion being two-dimensional, satisfies the relation

$$\frac{P_a}{P_{90^\circ}} = \frac{(4+\pi)\sin a}{4+\pi\sin a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

where a can have any value between 0° and 90°. If we compare this with experiment we must choose the results obtained with plates of large aspect ratio, since the theory refers to the extreme case of infinite aspect ratio. In practice, the aspect ratio most in use is 6 : 1, and we have already shown (in Fig. 91) that the agreement is sufficiently close to afford some confidence in both the basis of the theory and the methods of experiment.

But the formula is too complicated for use in aeroplane theory. We notice, however, that in aeroplanes the surfaces which are mainly responsible for the possibility of safe flight—namely, the wings and the tail—are arranged in such a way that the angle of attack is small. In steady horizontal flight, *e.g.*, with propeller axis horizontal, only the main plane or wings have an angle of attack at all: the elevator, rudder, and also the fixed parts of the tail, are such that the direction of motion lies in their planes. This suggests that for aeroplane theory it is possible to use an approximate formula, which will at any rate yield a comparatively simple *a priori* method of investigating aeroplane problems.

If a is small, then equation (1) can, neglecting a^2, be written

$$\frac{P_a}{P_{90^\circ}} = \frac{4+\pi}{4}\sin a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that the air pressure is proportional to the sine of the angle of attack.

Since we shall deal only with small values of a, it is not necessary to particularise the relation between P_a and P_{90}, and we shall write

$$P_a \text{ varies as } \sin a, \quad \ldots \ldots \ldots \quad (3)$$

the constant of proportionality being determined by experiment for the particular surface used. If S is the area of the plate we shall, therefore, have

$$P_a = KSU^2 \sin a, \quad \ldots \ldots \ldots \quad (4)$$

where U is the resultant velocity, and K is a constant depending on aspect ratio, density, etc. It is useful to note that this means that the air pressure on a plane is proportional to the product of the normal velocity into the resultant velocity.

The distance of the centre of pressure from the centre of the plate is given by theory, § 173, as

$$\tfrac{3}{4}l \frac{\cos a}{4 + \pi \sin a}, \quad \ldots \ldots \ldots \quad (5)$$

where l is the width. This also agrees fairly well with experiment. But here, too, we can, in studying aeroplane phenomena, introduce a great simplification due to the fact that the wings and also the other planes have, in practice, large aspect ratios.

This means that the planes are, in practice, comparatively narrow. If we have any change in the angle of attack a, we have a shift in the position of the centre of pressure, and consequently a change in the moment about the centre of gravity of the machine. We shall neglect this change and assume that the centre of pressure, for any plane, is invariable. This is tantamount to assuming each plane to have negligible width, and we shall remember this in dealing with the aerodynamic effects.

In this way we have evolved an idealised type of aeroplane characterised by **narrow planes at small angles of attack**. The simplification thus imparted enables us to discuss the problem of the aeroplane *a priori*. It is true that the practical problem will differ considerably from the ideal one thus constructed. Nevertheless, we shall find that most of the essentials of aeroplane theory as applied to practical flight can be deduced from the consideration of the ideal aeroplane.

The student will now have no difficulty in accepting the further simplification that the air pressure (almost entirely a resistance) due to the body of the machine, the landing chassis, etc., is omitted, and that we shall take no account of the changes in the air effects due to the propeller. Experimental aerodynamics is at present so little advanced that the data necessary for the effective *a priori* discussion of some important aeroplane problems are not yet available, and we are reduced to such processes of abstraction if we are to gain any insight at all into the types of results to expect.

The consideration of "narrow planes at small angles of attack" is due to G. H. Bryan, who was able to anticipate much of modern aeroplane theory and practice in spite of the artificial nature of the idealised aeroplane he considered. We shall from time to time indicate the results of experiment as applied to practical aeroplane problems.

191. **Longitudinal Steady Motion.**—Let us consider first the problem of longitudinal steady motion and stability. We shall take the case where steady horizontal flight with propeller axis horizontal is obtained by leaving the elevator in its neutral position. This is what we call Normal Flight. Let S_1 in Fig. 106 be the area of the main plane (or the sum of the areas for biplanes, etc.), U the actual velocity, a the angle of attack. Then the air pressure is $R = KS_1U^2 \sin a$ in a direction

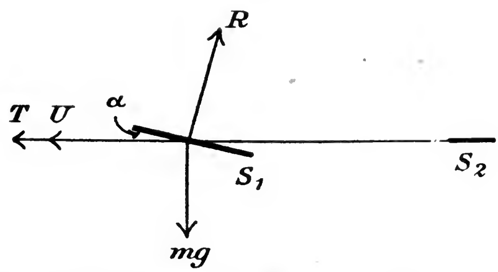

FIG. 106.—Longitudinal Steady Motion of Idealised Aeroplane.

normal to the plane. We assume that the centre of gravity of the whole aeroplane is at the centre of pressure. The conditions for steady flight are

$$mg = R \cos a, \; T = R \sin a,$$

i.e.

$$mg = KS_1U^2 \sin a \cos a, \; T = KS_1U^2 \sin^2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

192. **Changed Propeller Thrust.**—What is the steady motion for neutral elevator but changed thrust? Since the elevator is neutral, the motion must still be along the propeller axis, and the tail has no effect

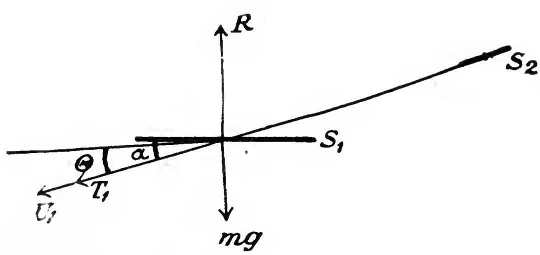

FIG. 107.—Steady Motion with Changed Propeller Thrust.

on the steady motion. If Θ is the angle that the direction of the velocity makes with the horizontal, measured positive downwards, Fig. 107, we have (for the same angle of attack, a)

$$mg \cos \Theta = R \cos a, \; T' + mg \sin \Theta = R \sin a,$$

where T' is the new thrust. If U' is the new velocity, we have

$$mg \cos \Theta = KS_1U'^2 \sin a \cos a, \; T' + mg \sin \Theta = KS_1U'^2 \sin^2 a \quad . \quad . \quad (7)$$

These two equations determine Θ and U', verifying that a change in the

thrust alone gives a change in the velocity of steady flight as well as in the orientation of the aeroplane in space (Chapter III., § 68).

By division we obtain

$$\frac{T' + mg \sin \Theta}{mg \cos \Theta} = \tan a = \frac{T}{mg},$$

T being the original thrust for horizontal motion ; hence

$$T' - T \cos \Theta = - mg \sin \Theta = - \frac{T}{\tan a} \sin \Theta,$$

so that

$$\sin (a - \Theta) = \frac{T'}{T} \sin a. \qquad (8)$$

We, therefore, deduce that when $T' > T$, Θ is negative, and that when $T'' < T$, Θ is positive. Thus an increase in the thrust gives climbing steady motion, whilst a decrease in the thrust gives a descending steady motion.

193. **Gliding.**—$T''=0$ gives $\Theta = a$, *i.e.* when the thrust is zero the steady motion is a glide at a gliding angle a, Fig. 108. For the

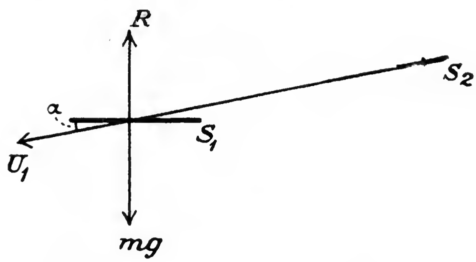

FIG. 108.—Gliding with Engine cut off.

idealised aeroplane, the gliding angle is the angle of attack of the main plane. The velocity of the glide is given by the fact that the air pressure balances the weight, so that

$$mg = KS_1 U'^2 \sin a \qquad (9)$$

But for the steady horizontal flight we had

$$mg = KS_1 U^2 \sin a \cos a \,;$$

hence

$$\frac{U'}{U} = \cos^{\frac{1}{2}} a, \qquad (10)$$

and the rate of vertical fall is

$$U' \sin a = U \sin a \cos^{\frac{1}{2}} a \qquad (11)$$

194. **Climbing.**—In the more general case we have from (6) and (7)

$$U' = U \cos^{\frac{1}{2}} \Theta \qquad (12)$$

and the rate of fall is

$$U \sin \Theta \cos^{\frac{1}{2}} \Theta, \qquad (13)$$

Θ being given by the equation (8). If Θ is negative, we get the rate of

climb. Taking the case where the change in T is comparatively small, so that Θ is small, we get by (8)

$$\Theta = -\frac{\delta T}{T}\tan a,$$

and the rate of climb is

$$\frac{U\delta T}{T}\tan a, \quad \ldots \ldots \ldots \ldots \quad (14)$$

whence obvious conditions for a good climber are obtained.

195. **The Elevator.**—Let us now suppose that the elevator comes into play. It is assumed that the whole of the horizontal tail plane acts as an elevator, although this is not the case in practice, where a portion of this plane is rigidly attached to the machine and only the remainder is used as elevator. Let a_1 be the angle that the main plane makes with the propeller axis (called a in §§ 191–94), a_2 the angle that the elevator makes with this axis, both measured positive as in the figure; let T, the thrust of the propeller, remain unaltered, and let Θ be the angle that T makes with the horizontal, measured positive downwards.. Since the propeller thrust and the weight pass through the centre of gravity, and it is assumed

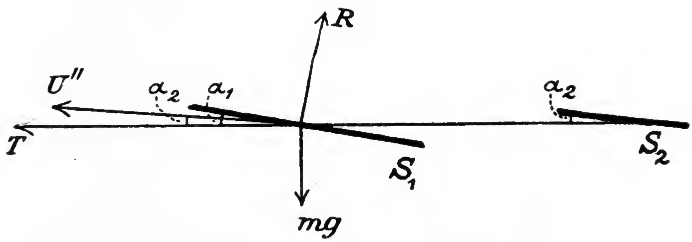

FIG. 109.—Steady Motion with Elevator in Use.

that the air pressure on the main plane also passes through the centre of gravity (the motion of the centre of pressure being neglected), it follows that the direction of motion of the machine in steady motion must be tangential to the elevator. Hence the velocity U'' makes an angle a_2 with the propeller axis, and, therefore, an angle $a_1 - a_2$ with the main plane. The conditions for statical equilibrium are

$$T + mg\sin\Theta = KS_1U''^2\sin(a_1 - a_2)\sin a_1,$$
$$mg\cos\Theta = KS_1U''^2\sin(a_1 - a_2)\cos a_1,$$

the pressure on the main plane being

$$KS_1U''^2\sin(a_1 - a_2).$$

We get

$$\frac{T + mg\sin\Theta}{mg\cos\Theta} = \tan a_1 = \frac{T}{mg}.$$

Hence we must conclude that $\Theta = 0$. The orientation of the machine is unaltered.

But the velocity is given by the fact that U was the velocity when $a_2 = 0$. This means that

$$T = KS_1U''^2\sin(a_1 - a_2)\sin a_1 = KS_1U^2\sin^2 a_1,$$
$$mg = KS_1U''^2\sin(a_1 - a_2)\cos a_1 = KS_1U^2\sin a_1\cos a_1.$$

Both equations are satisfied if we put

$$U''^2 = \frac{\sin a_1}{\sin (a_1 - a_2)} U^2.$$

Thus the aeroplane flies at an angle a_2 to the horizon with a modified velocity. If the elevator is turned down ($a_2 +$), the aeroplane climbs; if it is turned up ($a_2 -$), the aeroplane descends, in the new steady motion. (See Fig. 109.)

The student can investigate for himself the case where the propeller thrust is altered and the elevator is used. He will find that the orientation of the machine is affected.

196. **Longitudinal Stability.**—We shall now examine the condition for longitudinal stability, commencing with the case of normal flight, *i.e.* steady horizontal flight with the propeller axis horizontal and elevator in a neutral position. We neglect the effect of the air pressure changes due to the propeller, and take the case of a monoplane.

Let S_1 (Fig. 110) be the area of the main plane, K the air-pressure constant as already used. If the air pressure constant for the tail plane is K', multiply the area of the tail plane by K'/K and let S_2 be its

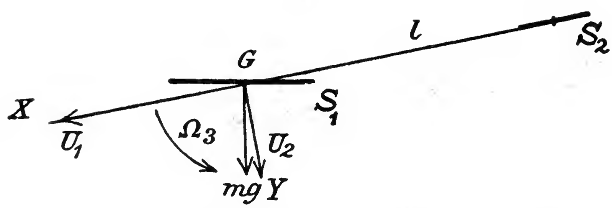

FIG. 110.—Longitudinal Stability of Idealised Aeroplane.

reduced area. If U_1, U_2 are the velocity components of the centre ot gravity measured along the directions of GX, GY, and Ω_3 is the angular velocity, then the velocity of the mid-point of the tail plane has components U_1, $U_2 - l\Omega_3$, where l is its distance *behind* the centre of gravity. Remembering that the air pressure on a plane is proportional to the product of the resultant velocity into the normal velocity, we get air pressures

$$\left.\begin{aligned} &KS_1(U_1^2 + U_2^2)^{\frac{1}{2}}(U_1 \sin a + U_2 \cos a) \text{ normal to } S_1, \\ &KS_2(U_1^2 + \overline{U_2 - l\Omega_3}^2)^{\frac{1}{2}}(U_2 - l\Omega_3) \quad \text{ normal to } S_2. \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

The rotation Ω_3 has, as such, no effect on these pressures, as we assume the planes to be narrow, so that any rotation produces no appreciable difference between the motions of the *leading* and *trailing* edges of any plane. We thus get

$$\left.\begin{aligned} mR_1 &= KS_1(U_1^2 + U_2^2)^{\frac{1}{2}}(U_1 \sin a + U_2 \cos a) \sin a, \\ mR_2 &= KS_1(U_1^2 + U_2^2)^{\frac{1}{2}}(U_1 \sin a + U_2 \cos a) \cos a \\ &\quad + KS_2(U_1^2 + \overline{U_2 - l\Omega_3}^2)^{\frac{1}{2}}(U_2 - l\Omega_3), \\ CG_3 &= - lKS_2(U_1^2 + \overline{U_2 - l\Omega_3}^2)^{\frac{1}{2}}(U_2 - l\Omega_3), \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

where m is the total mass and C is the moment of inertia about an axis through the centre of gravity perpendicular to the vertical plane of motion.

In the steady motion let $U_1 = U$, $U_2 = 0$, $\Omega_3 = 0$. Then in a small deviation from steady motion we take

$$U_1 = U + u_1,\ U_2 = u_2,\ \Omega_3 = \omega_3,$$

where u_1, u_2, ω_3 are small; these are, in fact, the quantities u_1, u_2, ω_3 used in the general theory, Chapter III. We easily deduce

$$\left.\begin{aligned} mR_1 &= KS_1U^2\sin^2 a\left(1+\frac{2u_1}{U}+\frac{u_2}{U}\cot a\right),\\ mR_2 &= KS_1U^2\sin a\cos a\left(1+\frac{2u_1}{U}+\frac{u_2}{U}\cot a\right)\\ &\quad + KS_2U(u_2 - l\omega_3),\\ CG_3 &= -lKS_2U(u_2 - l\omega_3).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

Hence

$$mR_{10} = KS_1U^2\sin^2 a = T,\ mR_{20} = KS_1U^2\sin a\cos a = mg,\ G_{30} = 0, \quad . \quad . \quad (18)$$

the conditions of steady flight. We also get, after reduction,

$$\left.\begin{aligned} a_x &= \frac{2g\tan a}{U^2}, & b_x &= \frac{g}{U^2}, & f_x &= 0,\\ a_y &= \frac{2g}{U^2}, & b_y &= \frac{g\cot a}{U^2}+\frac{gS_2}{S_1U^2\sin a\cos a}, & f_y &= -\frac{gS_2l}{S_1U^2\sin a\cos a},\\ a_3 &= 0, & b_3 &= -\frac{lKS_2}{C}, & f_3 &= \frac{l^2KS_2}{C}.\end{aligned}\right\} \quad (19)$$

197. Substituting in the determinantal equation 106 (Chapter III.) with Θ_0 zero, we get

$$\begin{vmatrix} \lambda+\dfrac{2g\tan a}{U^2}, & \dfrac{g}{U^2}, & -\dfrac{g}{U^2}\\ \dfrac{2g}{U^2}, & \lambda+\dfrac{g\cot a}{U^2}+\dfrac{gS_2}{S_1U^2\sin a\cos a}, & \left(1-\dfrac{gS_2l}{S_1U^2\sin a\cos a}\right)\lambda\\ 0, & -\dfrac{lKS_2}{C}, & \lambda^2+\dfrac{l^2KS_2}{C}\lambda \end{vmatrix} = 0. \quad (20)$$

If we work out this determinant, we find it to be

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

where

$$\left.\begin{aligned} A_1 &= 1,\\ B_1 &= \frac{g}{U^2}(2\tan a + \cot a) + \frac{l^2KS_2}{C} + \frac{gS_2}{S_1U^2\sin a\cos a},\\ C_1 &= \frac{2g^2S_2}{S_1U^4\cos^2 a} + \frac{gl^2KS_2}{CU^2}(2\tan a + \cot a) + \frac{lKS_2}{C},\\ D_1 &= \frac{2glKS_2\tan a}{CU^2},\\ E_1 &= \frac{2g^2lKS_2}{CU^4}.\end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

It is first of all necessary that all these coefficients shall have the same sign. Since A_1 is $+1$, it follows that B_1, C_1, D_1, E_1 must all be positive, whence we deduce that in the idealised aeroplane the tail plane must be *behind* the main plane, *i.e.* l must be positive. If this is the case, we get all the coefficients positive.

198. **The Condition of Longitudinal Stability.**—There remains the condition

$$H_1 \equiv B_1C_1D_1 - E_1B_1^2 - A_1D_1^2 > 0.$$

We can, of course, substitute in H_1 the values of the coefficients, and obtain the general condition of stability for the case we are considering. This would, however, be of little use, since the result would be far too complicated to yield any insight into the physical meaning of the condition. We now make use of certain facts about an aeroplane that enable us to simplify the algebra and at the same time give a physical meaning to the condition.

In an aeroplane the angle of attack a is small—it is the smallness of this angle that makes flight possible with the engines at present available. Let us take a about $\frac{1}{10}$th of a radian. Also we can, *e.g.*, make U about 100 feet per second, l about 30 or 40 feet, C about 30 times the mass, using feet in the radius of gyration. We find that in H_1 the term $A_1D_1^2 (\equiv D_1^2)$ can be neglected in comparison with $B_1C_1D_1$ or $E_1B_1^2$, and the condition $H_1 > 0$ can be written in the simplified form

$$B_1C_1D_1 > E_1B_1^2,$$

i.e.

$$\frac{D_1}{E_1}C_1 > B_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad (22)$$

The condition of longitudinal stability is, therefore, approximately

$$\frac{U^2 \tan a}{g}\left\{\frac{2g^2S_2}{S_1U^4\cos^2 a} + \frac{gl^2KS_2}{CU^2}(2\tan a + \cot a) + \frac{lKS_2}{C}\right\}$$
$$> \frac{g}{U^2}(2\tan a + \cot a) + \frac{gS_2}{S_1U^2\sin a\cos a} + \frac{l^2KS_2}{C},$$

in which we can approximate further by omitting terms which are small by virtue of the construction of the aeroplane. If we put $C = mk_3^2$ and use the relation $mg = KS_1U^2 \sin a \cos a$, we find that the condition becomes

$$\frac{lU^2\tan a}{k_3^2 g}\left(1 + \frac{2gk_3^2\sin a}{U^2 l\cos a} + \frac{2gl\sin a}{U^2\cos a}\right) > 1 + \frac{S_1}{S_2}(2\sin^2 a + \cos^2 a).$$

To one degree of approximation we can omit the second and third terms in the bracket on the left hand side, and can write $1 + S_1/S_2$ on the right hand side. We get

$$\frac{U^2}{g} > \frac{k_3^2}{l\tan a}\left(1 + \frac{S_1}{S_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This is Bryan's condition, equivalent to the condition obtained by Lanchester by means of the phugoid theory.

Using the relation between the weight W and the construction of the machine, we easily deduce as the approximate condition of stability

$$\frac{l}{g} > \frac{k_3^2}{W}\frac{KS_1}{S_2}(S_1 + S_2), \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad (24)$$

a more convenient form since it involves only the data of the planes and their distance apart, and not the velocity or the angle of attack. We see that the physical interpretation is that *the tail plane must be sufficiently far behind the main plane.*

199. **The Oscillations.**—Our approximate analysis suggests a method for arriving at a close presentation of the actual motion, when, by any means, the motion of the aeroplane is slightly different from the steady motion suitable to its shape and thrust, this steady motion being assumed horizontal with propeller axis horizontal, *i.e.* normal flight.

The numbers B_1, C_1 in the equation

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0, \quad \ldots \ldots \quad (25)$$

($A_1 = 1$) are so much bigger than D_1, E_1, that we can suppose two values of λ to be given by the equation

$$\lambda^2 + B_1\lambda + C_1 = 0. \quad \ldots \ldots \ldots \quad (26)$$

Thus in an actual machine (§ 206) the equation for λ was found to be

$$(U\lambda)^4 + 9{\cdot}44\,(U\lambda)^3 + 25{\cdot}44\,(U\lambda)^2 + 3{\cdot}08\,(U\lambda) + 1{\cdot}92 = 0. \quad \ldots \quad (27)$$

If we let $U\lambda_1$ be a value given by the approximate equation

$$(U\lambda_1)^2 + 9{\cdot}44\,(U\lambda_1) + 25{\cdot}44 = 0,$$

and then write $\lambda = \lambda_1 + \lambda'$, when λ' is to be a correction so as to satisfy the equation (27), we find that approximately

$$U\lambda' = -\frac{3{\cdot}08\,U\lambda_1 + 1{\cdot}92}{4(U\lambda_1)^3 + 28{\cdot}32\,(U\lambda_1)^2 + 50{\cdot}88\,(U\lambda_1)},$$

which reduces to

$$\lambda' = -\frac{3{\cdot}08\,(U\lambda_1) + 1{\cdot}92}{38{\cdot}23\,(U\lambda_1) + 240{\cdot}15},$$

where

$$U\lambda_1 = -\,4{\cdot}72 \pm \iota\sqrt{3{\cdot}16},$$

and it is evident that λ'/λ_1 is a small quantity.

We, therefore, write

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 \equiv \left(\lambda^2 + B_1\lambda + C_1\right)\left(\lambda^2 + F_1\lambda + \frac{E_1}{C_1}\right) = 0 \quad . \quad . \quad (28)$$

where F_1 is to be chosen so as to get a good approximation. If we use $\lambda^2 + E_1/C_1 = 0$, and substitute in the original biquadratic equation so as to get a more correct equation for the two roots given by the second factor, we get

$$\frac{E_1{}^2}{C_1{}^2} - \frac{B_1E_1}{C_1}\lambda + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

i.e.

$$C_1\lambda^2 + \frac{C_1D_1 - E_1B_1}{C_1}\lambda + E_1 + \frac{E_1{}^2}{C_1{}^2} = 0.$$

We can neglect $E_1{}^2/C_1{}^2$ compared to E_1, and so we get

$$C_1\lambda^2 + \frac{C_1D_1 - E_1B_1}{C_1}\lambda + E_1 = 0.$$

Hence we say that, *subject to certain limitations*, we can write

$$\lambda^4+B_1\lambda^3+C_1\lambda^2+D_1\lambda+E_1 \equiv \left(\lambda^2+B_1\lambda+C_1\right)\left(\lambda^2+\frac{C_1D_1-E_1B_1}{C_1{}^2}\lambda+\frac{E_1}{C_1}\right) = 0 \quad . \quad . \quad (29)$$

The limitations are found by comparing the coefficients of the various

powers of λ in the two assumed identical forms of the biquadratic. We must have approximate equality between the pairs of coefficients of λ^3 and λ^2, *i.e.*

$$\left.\begin{aligned} &B_1 \text{ and } B_1 + \frac{D_1}{C_1} - \frac{E_1B_1}{C_1^2}, \\ &C_1 \text{ and } C_1 + B_1\frac{C_1D_1 - E_1B_1}{C_1^2} + \frac{E_1}{C_1}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence D_1/B_1C_1, E_1/C_1^2 must be small and B_1/C_1 must not be large. The limits adopted by Bairstow (to whom this investigation is due) are that

$$\left.\begin{aligned} &\frac{D_1}{B_1C_1}, \ \frac{E_1}{C_1^2} \text{ must not be greater than } \tfrac{1}{20}, \\ \text{and} \quad &\frac{B_1}{C_1} \text{ must not be greater than } 1. \end{aligned}\right\} \quad (31)$$

If we include the coefficient A_1, the conditions are

$$\frac{A_1D_1}{B_1C_1}, \ \frac{A_1E_1}{C_1^2} \not> \tfrac{1}{20}, \ \frac{B_1}{C_1} \not> 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

These conditions are found to be satisfied for most modern types of aeroplanes.

200. **The Short Period Oscillation.**—We can now determine the kind of motion the aeroplane performs. The quantities B_1, C_1 are always positive—they are positive for the ideal machine, and are found to be positive in all actual machines. Hence the first factor in (29) cannot give rise to instability. In practice B_1, C_1 are fairly large, and we get oscillations of short period $\dfrac{2\pi}{U\sqrt{C_1 - B_1^2/4}}$ and quickly damped, the amplitude being, in fact, proportional to $e^{-\frac{UB_1t}{2}}$. In the machine already mentioned we get a period of $\dfrac{2\pi}{\sqrt{3\cdot16}}$ seconds, and amplitude proportional to $e^{-4\cdot72.t}$. The period is about $3\frac{1}{2}$ seconds, and the amplitude reduces to one-half in about one-sixth of a second.

201. **The Long Period Oscillation; Instability.**—The important factor in (29) is the second one. The two quantities E_1/C_1 and $(C_1D_1 - E_1B_1)/C_1^2$ must both be positive. The second condition is, in fact, the stability condition (22) already obtained by means of $H_1 > 0$. In practice, both these quantities are small, even if they are both positive. Thus in a stable machine we get an oscillation with a long period approximately $2\pi \Big/ U\sqrt{\dfrac{E_1}{C_1}}$, and slowly damped, the amplitude being proportional to $e^{-U\frac{C_1D_1 - E_1B_1}{2C_1^2}t}$. In the above machine the period is about 23 seconds, and the amplitude reduces to one half in about 20 seconds.

Instability can occur through E_1 not being positive, or through C_1D_1 not being greater than E_1B_1. If E_1 is negative we do not get an oscillation at all, and the machine departs continually from the steady motion. If E_1 is positive, but the condition $C_1D_1 > E_1B_1$ is not satisfied through D_1 not being big enough, the machine begins to oscillate, but the amplitude increases, and ultimately the steady motion is departed from.

This second case, therefore, produces an increasingly violent long period oscillation. In either case, we can no longer use the approximation assumed in § 199, and the actual motion must be determined by a discussion of the original equations of motion.

202. **Non-Horizontal Flight.**—If the direction of flight (still along the propeller axis) is inclined to the horizontal, we do not put Θ_0 zero in equations (105), Chapter III. The conditions of steady flight are

$$mg\cos\Theta_0 = KS_1U^2\sin\alpha\cos\alpha,$$
$$T + mg\sin\Theta_0 = KS_1U^2\sin^2\alpha.$$

The x, y derivatives now have $g\cos\Theta_0$ in the numerators instead of g, and if we substitute in (106), Chapter III., and divide the first two rows by $\cos\Theta_0$, we get

$$\begin{vmatrix} \dfrac{\lambda}{\cos\Theta_0}+\dfrac{2g\tan\alpha}{U^2}, & \dfrac{g}{U^2}, & -\dfrac{g}{U^2} \\ \dfrac{2g}{U^2} & \dfrac{\lambda}{\cos\Theta_0}+\dfrac{g\cot\alpha}{U^2}+\dfrac{gS_2}{S_1U^2\sin\alpha\cos\alpha}, & \left(\dfrac{1}{\cos\Theta_0}-\dfrac{gS_2l}{S_1U^2\sin\alpha\cos\alpha}\right)\lambda+\dfrac{g}{U^2}\tan\Theta_0 \\ 0, & -\dfrac{lKS_2}{C}, & \lambda^2+\dfrac{l^2KS_2}{C}\lambda \end{vmatrix} = \quad (33)$$

U being the velocity along the axis of the propeller. If we expand the determinant, we get

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

where $A_1 = \sec^2\Theta_0$. A comparison of (33) with (20) reveals the fact that the term $\frac{g}{U^2}\tan\Theta_0$ in the last element of the second row in (33) only affects D_1 and E_1. It, therefore, follows that B_1, C_1 are essentially positive for a tail behind the main plane. But we get

$$\left.\begin{aligned} D_1 &= \frac{glKS_2}{CU^2\cos\Theta_0}(2\tan\alpha+\tan\Theta_0),\\ E_1 &= \frac{2g^2lKS_2}{CU^4\cos\alpha\cos\Theta_0}\cos(\alpha-\Theta_0). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

There is no difficulty about E_1, since, in fact, Θ_0 is generally quite moderate. But D_1 can become negative. It vanishes when $\tan\Theta_0 = -2\tan\alpha$, and for greater negative values of Θ_0, D_1 is actually negative. If we refer to § 201, we see that this means a long period oscillation of increasing amplitude. Thus, if a machine is ascending at an angle $> \tan^{-1}(2\tan\alpha)$ (practically $> 2\alpha$), the stability is destroyed. In general, we see that the stability diminishes as the angle of ascent increases from zero.

This result is known as the **Harper Effect.** We cannot expect that the effect shall be reproduced exactly in actual flight, because we have idealised our aeroplane almost out of all recognition. The result is, nevertheless, of value as indicating a danger to be guarded against in practice.

203. **Experimental Determination of the Resistance Derivatives.**—It is obvious that the method of the idealised aeroplane can give results of only very limited application. The air pressure on the body of the machine, the camber, etc., introduce very important effects. Nevertheless, the investigation of the idealised aeroplane as here given, and as more

fully discussed by Bryan, has led to the successful attack of the general problem.

The process adopted in practice, is to deduce the derivatives required in the stability conditions from observations made on models in wind channels. It is not the purpose of this book to enter into a description of the practical side of these methods. We are only concerned with the dynamical theory and results.

Let us take the special case of the longitudinal stability of horizontal flight with propeller axis horizontal, there being symmetry about the vertical plane of motion.

204. **Derivatives due to u_1, u_2.**

a_x, a_y, a_3.—These derivatives are obtained by finding the rates of change of R_1, R_2, G_3, when U is changed. Let U become $U + u_1$. This new velocity is in the same direction as the old, and since U caused no tendency to turn, it follows that $U + u_1$ causes no tendency to turn; in fact, we have R_1 and R_2 proportional to U_1^2 and G_3 zero, no matter how U is changed. Hence $a_3 = 0$, and

$$\frac{1}{R_1}\frac{\partial R_1}{\partial U} = \frac{2}{U}, \quad \frac{1}{R_2}\frac{\partial R_2}{\partial U} = \frac{2}{U}.$$

It follows that

$$a_x = 2R_{10}/U^2, \quad a_y = 2R_{20}/U^2, \quad a_3 = 0. \qquad (35)$$

b_x, b_y, b_3.—To find the derivatives due to u_2, we note that a small velocity u_2 really gives a resultant velocity $(U^2 + u_2^2)^{\frac{1}{2}}$ along a different direction from that of U, this direction being at an angle $\tan^{-1} u_2/U$ below the direction of U. This is an increase in the angle of attack by the amount u_2/U when u_2 is infinitesimal. If a is the angle of attack, we, therefore, get

$$b_x = \frac{1}{U^2}\left(\frac{\partial R_1}{\partial a}\right), \quad b_y = \frac{1}{U^2}\left(\frac{\partial R_2}{\partial a}\right), \quad b_3 = \frac{1}{U^2}\left(\frac{\partial G_3}{\partial a}\right), \qquad (36)$$

the brackets denoting that we imagine first that R_1, R_2, G_3 are written down as functions of a, then differentiated, and then we substitute the value of the angle of attack in the steady flight.

In practice it is usual to measure the air pressure force in terms of the lift L and drag D perpendicular to, and backwards along, the direction of flight, or perpendicular to and along the direction of the relative wind. To express b_x, b_y in terms of L and D, we put

$$mR_1\left(a + \frac{u_2}{U}\right) = D\left(a + \frac{u_2}{U}\right)\cos\frac{u_2}{U} - L\left(a + \frac{u_2}{U}\right)\sin\frac{u_2}{U},$$

$$mR_2\left(a + \frac{u_2}{U}\right) = D\left(a + \frac{u_2}{U}\right)\sin\frac{u_2}{U} + L\left(a + \frac{u_2}{U}\right)\cos\frac{u_2}{U}.$$

When u_2 is infinitesimal, these equations become

$$mR_1\left(a + \frac{u_2}{U}\right) = D(a) + \frac{u_2}{U}\left[\left(\frac{\partial D}{\partial a}\right) - L(a)\right],$$

$$mR_2\left(a + \frac{u_2}{U}\right) = L(a) + \frac{u_2}{U}\left[\left(\frac{\partial L}{\partial a}\right) + D(a)\right].$$

We, therefore, get

$$b_x = \frac{1}{mU^2}\left[\left(\frac{\partial D}{\partial a}\right) - L(a)\right], \quad b_y = \frac{1}{mU^2}\left[\left(\frac{\partial L}{\partial a}\right) + D(a)\right], \qquad (37)$$

in which we suppose L and D expressed as functions of a, differentiated

where necessary, and then evaluated with the value of a in the steady flight.

The derivative b_3 is calculated direct from observation.

205. **Derivatives due to ω_3.**

f_x, f_y, f_3.—The derivatives due to ω_3 arise from the fact that if there is a slight rotation, then the back of the machine experiences additional pressure. If l is the greatest length of the machine, we see that the effect of ω_3 cannot be greater than that of changing the direction of the resultant velocity by an angle $l\omega_3/U$. It follows that f_x, f_y cannot be numerically greater than l times b_x, b_y respectively. In practice they are found to be considerably less. Now in the determinantal equation (106), Chapter III., f_y occurs in conjunction with U_{10}/U, which is unity, whilst f_y is small. Hence f_y can be neglected. Further, since $a_3 = 0$, f_x occurs only in D_1 (the coefficient of λ), which is

$$\begin{vmatrix} a_x, & b_x, & f_x \\ a_y, & b_y, & 1 \\ 0\ , & b_3, & f_3 \end{vmatrix},$$

since $\Theta_0 = 0$, and $U_{10} = 0$; we have put $a_3 = 0$, and neglected f_y.

We see that f_x occurs in the product $f_x a_y b_3$, compared with a term like $a_x b_3$; *i.e.* $f_x a_y$ compared to a_x, *i.e.* $f_x R_{20}$ compared to R_{10}, since $a_x/a_y = R_{10}/R_{20}$. But R_{20}/R_{10} has as its maximum value a number like 15. Hence f_x occurs in such a way that it has to be compared to a quantity like $\frac{1}{15}$, and this is also a comparatively large quantity. Thus we can neglect f_x also.

We are, therefore, left with only f_3. To determine this derivative an experiment is performed which is in itself an interesting case of a problem in resisted motion. The model is set up in the wind channel, but is supported by means of springs in such a way that the effect of a wind is to make it oscillate about an axis perpendicular to the plane of symmetry. Suppose that this axis is at a distance l_1 below the centre of gravity. The couple due to the wind is $m(G_3 + l_1 R_1)$. We can clearly write

$$R_1 = R_{10} + U(a_x u_1 + f_x \omega_3), \quad G_3 = G_{30} + U(a_3 u_1 + f_3 \omega_3),$$

where $u_1 = l_1 \omega_3$ and ω_3 is the angular velocity supposed small. The additional couple due to any oscillation of the model is

$$m(G_3 - G_{30} + l_1 R_1 - l_1 R_{10}),$$

and the equation of motion is, therefore,

$$I_1 \frac{d^2\theta}{dt^2} + mU(l_1^2 a_x + l_1 a_3 + l_1 f_x + f_3)\frac{d\theta}{dt} + p^2\theta = 0, \quad . \quad . \qquad (38)$$

where I_1 is the moment of inertia about the axis used, θ is the angular displacement, so that $\omega_3 = d\theta/dt$, and p^2 is the elastic coefficient given by the springs, etc. If p is large enough, we get a damped oscillation in which the damping is defined by the exponential

$$e^{-\frac{mU}{2I_1}(l_1^2 a_x + l_1 a_3 + l_1 f_x + f_3)t}.$$

The time taken till the amplitude is reduced by a certain fraction, say one half, is observed, and thus the quantity

$$l_1^2 a_x + l_1 a_3 + l_1 f_x + f_3$$

is determined. Since a_x and a_3 are known, we thus find the value of

$$l_1 f_x + f_3.$$

If the experiment is repeated with a different depth l_2 of the axis below the centre of gravity, we determine, in the same way, the value of

$$l_2 f_x + f_3.$$

It follows that we can deduce f_3. Of course, f_x is also found, but, as already pointed out, it is not required in the discussion of stability.

206. **Some Experimental Results.**—To illustrate the mathematics, we quote the values of the derivatives as deduced by these methods in the case of one or two machines experimented on at the National Physical Laboratory.

In one machine of weight 1,300 lbs., made so as to travel at the rate of fifty-five miles per hour in a horizontal direction (the wing area being 300 sq. ft., and $k_3^2 = 25$), it was found that the following values could be used:

$$Ua_x = 0{\cdot}14, \qquad Ub_x = -0{\cdot}19,$$
$$Ua_y = 0{\cdot}80, \qquad Ub_y = 2{\cdot}89,$$
$$Ub_3 = -0{\cdot}106, \qquad Uf_3 = 8{\cdot}4.$$

For stability in horizontal flight the equation for λ becomes (putting $\Theta_0 = 0$)

$$(U\lambda)^4 + 11{\cdot}4\,(U\lambda)^3 + 33{\cdot}6\,(U\lambda)^2 + 5{\cdot}72\,(U\lambda) + 2{\cdot}72 = 0.$$

Since all the coefficients are positive, the only condition for stability is $H_1 = 0$, and in this case H_1 is found to be positive. The machine is, therefore, stable for the assumed steady motion.

For gliding stability, we put $\Theta_0 =$ the angle of glide. The equation for λ is

$$(U\lambda)^4 + 11{\cdot}4\,(U\lambda)^3 + 33{\cdot}6\,(U\lambda)^2 + 6{\cdot}40\,(U\lambda) + 2{\cdot}72 = 0\,;$$

once again all the coefficients are positive, and H_1 is positive and greater than before. The stability is increased as already suggested.

In another machine, of the Bleriot type, the values of the derivatives were

$$Ua_x = 0{\cdot}0935, \qquad Ub_x = -0{\cdot}152,$$
$$Ua_y = 0{\cdot}672, \qquad Ub_y = 2{\cdot}43,$$
$$Ub_3 = -0{\cdot}0884, \qquad Uf_3 = 7.$$

The equation for λ in the case of horizontal steady flight is

$$(U\lambda)^4 + 9{\cdot}44\,(U\lambda)^3 + 25{\cdot}44\,(U\lambda)^2 + 3{\cdot}08\,(U\lambda) + 1{\cdot}92 = 0.$$

This is, in fact, the machine referred to above, § 199. It is stable, since all the coefficients are positive, and H_1 is positive.

207. **Lateral Stability.**—The discussion of lateral stability is more difficult in the *a priori* method used by Bryan, since we have to find air pressures for planes moving in a more general manner than has yet been adequately investigated. It will, nevertheless, pay us to work through the necessary algebra, because of the manner in which the results suggest the practical methods used to ensure lateral stability, and also as an introduction to the theory of the propeller.

The method is only justifiable because of its success. There is little reason for making the fundamental assumption, which is, that if we take

any element of surface dS which contributes air pressure, then we can use for this element the expression

$$(KdS) \times (\text{resultant velocity}) \times (\text{normal velocity})$$

for the air pressure on it, and we can take this to act in a direction normal to dS. This is tantamount to assuming that we can take any element dS to be part of a plane of large aspect ratio with the air round it endowed with two-dimensional motion.

Let us consider an element of surface dS at the point X, Y, Z. If U_1, U_2, U_3, Ω_1, Ω_2, Ω_3 define the motion of the aeroplane measured with respect to axes fixed in the body, than the velocity components of the element are, Chapter IV., § 103,

$$U_1 + \Omega_2 Z - \Omega_3 Y, \quad U_2 + \Omega_3 X - \Omega_1 Z, \quad U_3 + \Omega_1 Y - \Omega_2 X. \qquad (39)$$

If l, m, n are the direction cosines of the normal to the element (it will be

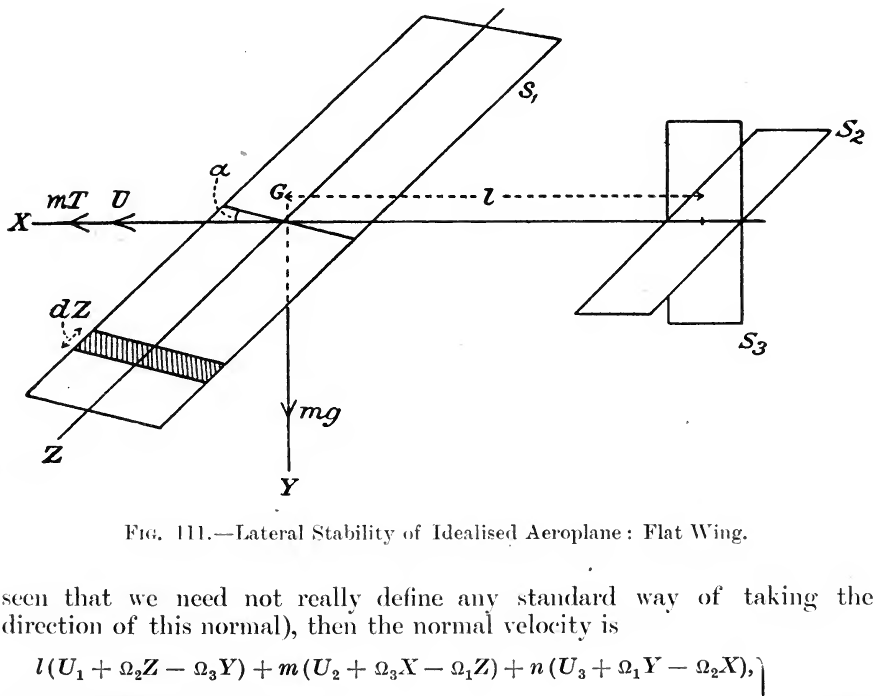

FIG. 111.—Lateral Stability of Idealised Aeroplane: Flat Wing.

seen that we need not really define any standard way of taking the direction of this normal), then the normal velocity is

$$\left.\begin{array}{l} l(U_1 + \Omega_2 Z - \Omega_3 Y) + m(U_2 + \Omega_3 X - \Omega_1 Z) + n(U_3 + \Omega_1 Y - \Omega_2 X), \\ \text{and the resultant velocity is} \\ \{(U_1 + \Omega_2 Z - \Omega_3 Y)^2 + (U_2 + \Omega_3 X - \Omega_1 Z)^2 + (U_3 + \Omega_1 Y - \Omega_2 X)^2\}^{\frac{1}{2}}. \end{array}\right\} \qquad (40)$$

The pressure on the element is, therefore, KdS into the product of these two velocities, and the components of force and couple contributed by this element are respectively

$$l,\ m,\ n,\ nY - mZ,\ lZ - nX,\ mX - lY \qquad (41)$$

times this element of pressure.

208. **Simplest Case.**—Let us now suppose that our idealised aeroplane, Fig. 111, consists of a very narrow wing S_1 of span $2a$, angle of

attack a, and a tail consisting of an elevator S_2 in the plane XZ and a rudder S_3 in the plane XY. We assume symmetry about the XY plane, which is also the vertical plane of steady motion. The latter is taken to be U along the X axis, the direction of this axis (the axis of the propeller) being also horizontal. We have already seen that the lateral and longitudinal stability are independent of one another in the symmetrical aeroplane with symmetrical steady motion. We, therefore, define disturbed motion (*i.e.* deviation from the appropriate steady motion) by writing

$$U_1 = U,\ U_2 = 0,\ U_3 = u_3,\ \Omega_1 = \omega_1,\ \Omega_2 = \omega_2,\ \Omega_3 = 0, \quad . \quad . \quad . \quad (42)$$

where u_3, ω_1, ω_2 are small quantities whose squares, etc., are assumed negligible.

The normal and resultant velocities for any element dS, X, Y, Z, l, m, n are respectively,

$$l(U + \omega_2 Z) + m(-\omega_1 Z) + n(u_3 + \omega_1 Y - \omega_2 X), \quad . \quad . \quad . \quad . \quad (43)$$

and

$$U + \omega_2 Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

the latter being taken to the first order of small quantities.

In order to simplify the calculations in this idealised problem, we shall assume that S_2 and S_3 are so small that we may neglect the squares of their dimensions. Another way of putting this is, that we shall take a sort of average for S_2 and S_3 by using for X, Y, Z the co-ordinates of their centres, *i.e.* $(-l, 0, 0)$ for both if, as is generally the case, the elevator and the rudder are at the same distance behind the wings. (The student should take care not to confuse this length l with the direction cosine l, which will soon drop out of the algebra.)

The direction cosines of the elevator S_2 are 0, 1, 0; the normal velocity for any element of its surface is, therefore, zero, and the pressure is zero. We can thus neglect S_2 altogether.

The rudder S_3 has direction cosines 0, 0, 1. Using the assumption just mentioned, we get, parallel to the Z axis inwards, a force

$$KS_3U(u_3 + \omega_2 l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

since for the mid-point of the rudder we have $Y = Z = 0$. The rudder, therefore, contributes a resistance force

$$\left.\begin{array}{l} KS_3U(u_3 + \omega_2 l) \text{ parallel to the } Z \text{ axis inwards,} \\ \text{and a resistance couple} \\ KS_3Ul(u_3 + \omega_2 l) \text{ about the } Y \text{ axis from } X \text{ to } Z. \end{array}\right\} \quad . \quad . \quad (46)$$

As for the wings, let us take them to be rectangular in shape, inclined at an angle a to the X axis, as in Fig. 111. If we take a strip dZ at distance Z from the origin, the area is $S_1 dZ/2a$, the co-ordinates of its centre can be taken to be $(0, 0, Z)$, and its direction cosines are $\sin a$, $\cos a$, 0. Hence the element of pressure is

$$\frac{KS_1dZ}{2a}(U + \omega_2 Z)\{(U + \omega_2 Z)\sin a - \omega_1 Z \cos a\},$$

i.e.

$$\frac{KS_1dZ}{2a}U^2 \sin a\left\{1 + \frac{2\omega_2}{U}Z - \frac{\omega_1 \cot a}{U}Z\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

to the first order of small quantities. This contributes components of force and couple:

$$\left.\begin{array}{l} 0 \text{ along the } Z \text{ axis,} \\ -\dfrac{KS_1}{2a} ZdZ \,.\, U^2 \sin\alpha\cos\alpha\left\{1+\dfrac{2\omega_2}{U}Z-\dfrac{\omega_1\cot\alpha}{U}Z\right\} \text{ about the } X \text{ axis, } Z \text{ to } Y, \\ \text{and} \\ \dfrac{KS_1}{2a} ZdZ \,.\, U^2 \sin^2\alpha \quad \left\{1+\dfrac{2\omega_2}{U}Z-\dfrac{\omega_1\cot\alpha}{U}Z\right\} \text{ about the } Y \text{ axis, } X \text{ to } Z. \end{array}\right\} \quad (48)$$

If we integrate for the whole of the wings, we get moments

$$\left.\begin{array}{l} -\dfrac{KS_1a^2}{3} U\sin\alpha\cos\alpha\,(2\omega_2-\omega_1\cot\alpha), \\ \dfrac{KS_1a^2}{3} U\sin^2\alpha\,(2\omega_2-\omega_1\cot\alpha). \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (49)$$

Taking the wings and rudder together, we get a force

$$KS_3U(u_3+\omega_2 l)$$

parallel to the Z axis inwards, and couples

$$-\frac{KS_1a^2}{3} U\sin\alpha\cos\alpha\,(2\omega_2-\omega_1\cot\alpha)$$

about the X axis, Z to Y, and

$$\frac{KS_1a^2}{3} U\sin^2\alpha\,(2\omega_2-\omega_1\cot\alpha)+KS_3Ul(u_3+\omega_2 l)$$

about the Y axis, X to Z.

Hence we get

$$\left.\begin{array}{rl} mR_3 = & KS_3U(u_3+\omega_2 l), \\ AG_1 = & \dfrac{KS_1a^2}{3} U\sin\alpha\cos\alpha\,(\omega_1\cot\alpha-2\omega_2), \\ BG_2 = - & \dfrac{KS_1a^2}{3} U\sin^2\alpha\,(\omega_1\cot\alpha-2\omega_2)+KS_3Ul(u_3+\omega_2 l), \end{array}\right\} \quad . \; . \quad (50)$$

where m is the mass of the aeroplane and A, B are the moments of inertia about the axes of X, Y respectively. If we write

$$mg = KS_1U^2\sin\alpha\cos\alpha$$

from the condition for steady motion, and

$$A = mk_1{}^2, \quad B = mk_2{}^2,$$

we get

$$\left.\begin{array}{lll} c_z = \dfrac{S_3g}{S_1U^2\sin\alpha\cos\alpha}, & d_z = 0, & e_z = \dfrac{S_3lg}{S_1U^2\sin\alpha\cos\alpha}, \\ c_1 = 0, & d_1 = \dfrac{a^2g\cot\alpha}{3k_1{}^2U^2}, & e_1 = -\dfrac{2a^2g}{3k_1{}^2U^2}, \\ c_2 = \dfrac{S_3lg}{S_1U^2k_2{}^2\sin\alpha\cos\alpha}, & d_2 = -\dfrac{ga^2}{3k_2{}^2U^2}, & e_2 = \dfrac{2ga^2\tan\alpha}{3k_2{}^2U^2}+\dfrac{S_3l^2g}{S_1U^2k_2{}^2\sin\alpha\cos\alpha}. \end{array}\right\} \quad (51)$$

209. **Spiral Instability.**—The determinantal equation (148), Chapter IV., with $\Theta_0 = 0$, $U_{10} = U$, $U_{20} = 0$, is

$$\begin{vmatrix} \lambda + c_z, & d_z\lambda + \frac{g}{U}, & e_z - 1 \\ c_1, & \lambda^2 + d_1\lambda, & e_1 \\ c_2, & d_2\lambda, & \lambda + e_2 \end{vmatrix} = 0, \quad . \quad . \quad . \quad (52)$$

in which we have assumed that the product of inertia F is negligible. If we expand and obtain

$$A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2 = 0,$$

we have $A_2 = 1$, and the other coefficients are again complicated expressions. But without working out the determinant completely, we can still show that E_2 is negative, so that the motion is unstable. We have

$$E_2 = \begin{vmatrix} c_z, & \frac{g}{U}, & e_z - 1 \\ c_1, & 0, & e_1 \\ c_2, & 0, & e_2 \end{vmatrix}$$

obtained by putting $\lambda = 0$ in the determinant (52). Hence

$$E_2 = \frac{g}{U}(e_1c_2 - c_1e_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Since c_1 is zero, we get $E_2 = \frac{g}{U}e_1c_2$, and we have seen that with the rudder behind the wing (and this is the general practice now) c_2 is positive and e_1 negative.

Can we assign a physical meaning to the instability thus discovered? It clearly arises from the fact that c_1 is zero, *i.e.* there is no moment about the X axis due to a small velocity along the Z axis. Since e_2 is multiplied by c_1, this has the effect of preventing the e_2 couple from opposing rotation about the Y axis. Also E_2 negative gives at least one real positive value of λ; hence the instability must consist of a continued rotation about the Y axis, and the machine tends to turn round and round, at the same time dropping, since it loses some of its velocity by transference of some energy into rotational energy. Thus we get a **spiral dive,** and the result is **spiral instability.**

210. **Prevention of Spiral Instability.**—To make E_2 positive it is necessary to have c_1 negative, since e_2 is clearly positive in our analysis, and, in fact, for any machine with a rudder behind. There can be no derivative c_1 in the case we have considered, because we have assumed all pressures to be normal to the planes, and the symmetry of the wings prevents the velocity u_3 giving a moment about an axis in the plane of symmetry, whilst the rudder itself cannot give any moment about the Y axis.

We must make some modification in our idealised aeroplane. In actual machines this is not always necessary, since the simplified assumptions we have made are not quite true, and c_1 can exist. It is, nevertheless, a very common practice to introduce one of two modifications (or both) in the shape of the wings, in order to increase the lateral stability, and we shall indicate the nature of these modifications.

It is necessary in the idealised aeroplane to make the wings such that a velocity u_3 destroys the symmetry in the moments about the X axis.

This can be done in one of two ways: (1) by means of a **dihedral angle,** (2) by means of **swept-back wings.** The former is the more usual, but the latter is a very prominent feature of several well-known types of machine.

211. **Dihedral Angle.**—Suppose that the wings consist of two planes making, with one another, an angle nearly, but not quite, 180° In Fig. 112 let $ABCD$ be one of these planes in the form of a narrow

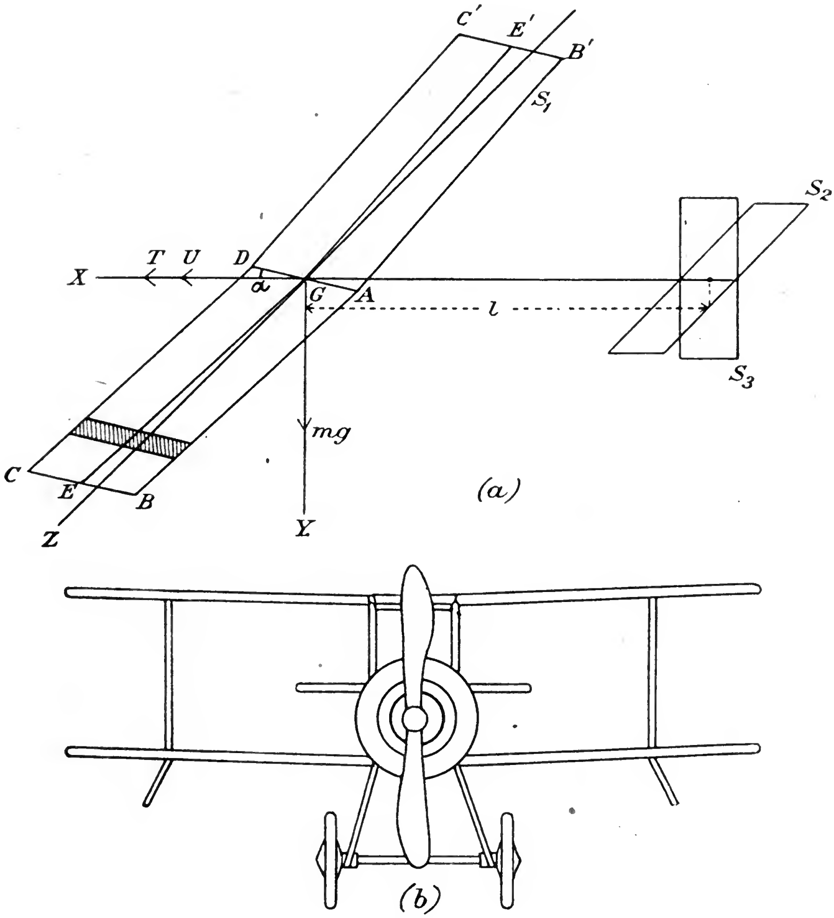

FIG. 112.—Lateral Stability: Dihedral Angle.

rectangle. In order to obtain zero moment about the Z axis in steady flight, we shall suppose that the wing $ABCD$ is such that the plane through the Z axis and the mid-line GE is perpendicular to the wing.

Let the equation of the plane $ABCD$ be

$$lX + mY + nZ = 0.$$

It cuts the plane $Z = 0$ in the line AD, whose equation is $lX + mY = 0$, *i.e.*

$$\frac{l}{m} = -\frac{Y}{X} = \frac{\sin a}{\cos a},$$

where a is the angle of attack defined by the angle between AD and the X axis. The equation of the plane is thus

$$m(X\tan a + Y) + nZ = 0, \quad \ldots \quad (54)$$

where, since $m\tan a$, m, n are direction cosines, we have $n^2 + m^2\sec^2 a = 1$. If now we take a point $(0, 0, \zeta)$ on the Z axis, the perpendicular through this point to the plane $ABCD$ has the equation

$$\frac{X}{m\tan a} = \frac{Y}{m} = \frac{Z-\zeta}{n}.$$

This cuts the plane $ABCD$ in the point where co-ordinates are easily shown to be

$$-mn\tan a\,.\,\zeta, \quad -mn\zeta, \quad m^2\sec^2 a\,.\,\zeta\,. \quad \ldots \quad (55)$$

If now we take a strip of the wing $ABCD$ defined by ζ, $\zeta + d\zeta$, its area is $S_1 d\zeta/2a$, where S_1 is the area of the *two* wings, and a is the value of ζ corresponding to the mid-point E of the end BC. The co-ordinates of the centre of this strip can be taken to be (55), and the direction cosines of the normal are

$$m\tan a, \quad m, \quad n. \quad \ldots \quad (56)$$

Substitute

$$U_1 = U, \quad U_2 = 0, \quad U_3 = u_3, \quad \Omega_1 = \omega_1, \quad \Omega_2 = \omega_2, \quad \Omega_3 = 0,$$

in (39), and we get for the velocity components of this element of area

$$U + m^2\sec^2 a\,.\,\zeta\omega_2, \quad -m^2\sec^2 a\,.\,\zeta\omega_1, \quad u_3 - mn\zeta\omega_1 + mn\tan a\,.\,\zeta\omega_2.$$

The normal velocity is

$$(U + m^2\sec^2 a\,.\,\zeta\omega_2)\,m\tan a - m^3\sec^2 a\,.\,\zeta\omega_1 + n(u_3 - mn\zeta\omega_1 + mn\tan a\,.\,\zeta\omega_2),$$

and the resultant velocity is, to the first order of the small quantities u_3, ω_1, ω_2

$$U + m^2\sec^2 a\,.\,\zeta\omega_2.$$

We get that the air pressure on the element is to the first order

$$\frac{KS_1U^2m\tan a}{2a}\left\{1 + \frac{n}{m\tan a}\frac{u_3}{U} - \cot a\,.\,\zeta\frac{\omega_1}{U} + (1 + m^2\sec^2 a)\,\zeta\frac{\omega_2}{U}\right\}d\zeta, \quad . \quad (57)$$

where K is a constant defining the pressure, depending on the air density, aspect ratio, etc. The components of force and couple due to this element of pressure are obtained by multiplying it by

$$\left.\begin{matrix} m\tan a, & m, & n, \\ -m\zeta, & m\tan a\,.\,\zeta, & 0. \end{matrix}\right\} \quad \ldots \quad (58)$$

Let us now take an exactly similar element on the other wing. We do this by changing the signs of n and ζ. If, then, we add up the effects of two symmetrical elements, we get components of force and couple:

$$\left.\begin{aligned}
&\frac{2KS_1m^2\tan^2 a\,.\,U^2d\zeta}{2a},\\
&\frac{2KS_1m^2\tan a\,.\,U^2d\zeta}{2a};\\
&\frac{2KS_1mn\tan a\,.\,U^2d\zeta}{2a}\left\{\frac{n}{m\tan a}\frac{u_3}{U} - \cot a\,.\,\zeta\frac{\omega_1}{U} + (1 + m^2\sec^2 a)\,\zeta\frac{\omega_2}{U}\right\},\\
-&\frac{2KS_1m^2\tan a\,.\,U^2\zeta d\zeta}{2a}\left\{\frac{n}{m\tan a}\frac{u_3}{U} - \cot a\,.\,\zeta\frac{\omega_1}{U} + (1 + m^2\sec^2 a)\,\zeta\frac{\omega_2}{U}\right\},\\
&\frac{2KS_1m^2\tan^2 a\,.\,U^2\zeta d\zeta}{2d}\left\{\frac{n}{m\tan a}\frac{u_3}{U} - \cot a\,.\,\zeta\frac{\omega_1}{U} + (1 + m^2\sec^2 a)\,\zeta\frac{\omega_2}{U}\right\},\\
&0;
\end{aligned}\right\} \quad (59)$$

thus verifying that there are no longitudinal derivatives due to lateral disturbances. Integrating, we get for both wings the following air forces and couples:

$$\left.\begin{aligned}
&KS_1m^2\tan^2 a\,.\,U^2,\\
&KS_1m^2\tan a\,.\,U^2;\\
&KS_1mn\tan a\,.\,U^2\left\{\frac{n}{m\tan a}\frac{u_3}{U}-\frac{a}{2\tan a}\frac{\omega_1}{U}+\frac{a}{2}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},\\
-&KS_1m^2\tan a\,.\,U^2\left\{\frac{an}{2m\tan a}\frac{u_3}{U}-\frac{a^2}{3\tan a}\frac{\omega_1}{U}+\frac{a^2}{3}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},\\
&KS_1m^2\tan^2 a\,.\,U^2\left\{\frac{an}{2m\tan a}\frac{u_3}{U}-\frac{a^2}{3\tan a}\frac{\omega_1}{U}+\frac{a^2}{3}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},\\
&0.
\end{aligned}\right\} \quad . \quad . \quad (60)$$

The equilibrium conditions are

$$\left.\begin{aligned}
KS_1m^2\tan a\;.\;U^2&=\text{weight},\\
KS_1m^2\tan^2 a\,.\,U^2&=\text{propeller thrust}.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The longitudinal equation of moments is automatically satisfied. The R_3 air force per unit mass and the G_1, G_2 couples per unit moment of inertia, in each case, are then found to be

$$\left.\begin{aligned}
R_3&=\frac{ng}{m}\left\{\frac{n}{m\tan a}\frac{u_3}{U}-\frac{a}{2\tan a}\frac{\omega_1}{U}+\frac{a}{2}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},\\
G_1&=-\frac{g}{k_1^2}\left\{\frac{an}{2m\tan a}\frac{u_3}{U}-\frac{a^2}{3\tan a}\frac{\omega_1}{U}+\frac{a^2}{3}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},\\
G_2&=\frac{g\tan a}{k_2^2}\left\{\frac{an}{2m\tan a}\frac{u_3}{U}-\frac{a^2}{3\tan a}\frac{\omega_1}{U}+\frac{a^2}{3}(1+m^2\sec^2 a)\frac{\omega_2}{U}\right\},
\end{aligned}\right\} \quad . \quad . \quad (62)$$

where k_1 is the radius of gyration about the X axis, k_2 about the Y axis.

Remembering the air forces and couples given by the rudder plane S_3 (see (46), § 208), we get in our present problem

$$c_z=\frac{S_3g}{S_1U^2m^2\tan a}+\frac{n^2g}{m^2\tan a\,.\,U^2},\quad d_z=-\frac{nga}{2m\tan a\,.\,U^2},\quad e_z=\frac{S_3lg}{S_1U^2m^2\tan a}+\frac{nga(1+m^2\sec^2 a)}{2mU^2},$$

$$c_1=-\frac{nga}{2mk_1^2U^2\tan a},\quad d_1=\frac{ga^2}{3k_1^2U^2\tan a},\quad e_1=-\frac{ga^2(1+m^2\sec^2 a)}{3k_1^2U^2},$$

$$c_2=\frac{S_3lg}{S_1U^2k_2^2m^2\tan a}+\frac{nga}{2mk_2^2U^2},\quad d_2=-\frac{ga^2}{3k_2^2U^2},\quad e_2=\frac{S_3l^2g}{S_1U^2k_2^2m^2\tan a}+\frac{ga^2\tan a(1+m^2\sec^2 a)}{3k_2^2U^2},$$

where l is the distance of the rudder behind the wings. If we substitute in the determinantal equation (52), we find that the coefficient E_2 in the biquadratic for λ is now

$$E_2=\frac{g}{U}(e_1c_2-c_1e_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$=\frac{g^2a^2S_3l}{3S_1U^4k_1^2k_2^2m^2\tan^2 a}\left\{\frac{3nl}{2ma}-\tan a(1+m^2\sec^2 a)\right\}; \quad . \quad . \quad . \quad (64)$$

and this can be made positive by making n large enough. We thus prevent spiral instability by making $\frac{3nl}{2ma} > 2 \tan a$, since $m^2 \sec^2 a = 1 - n^2$;

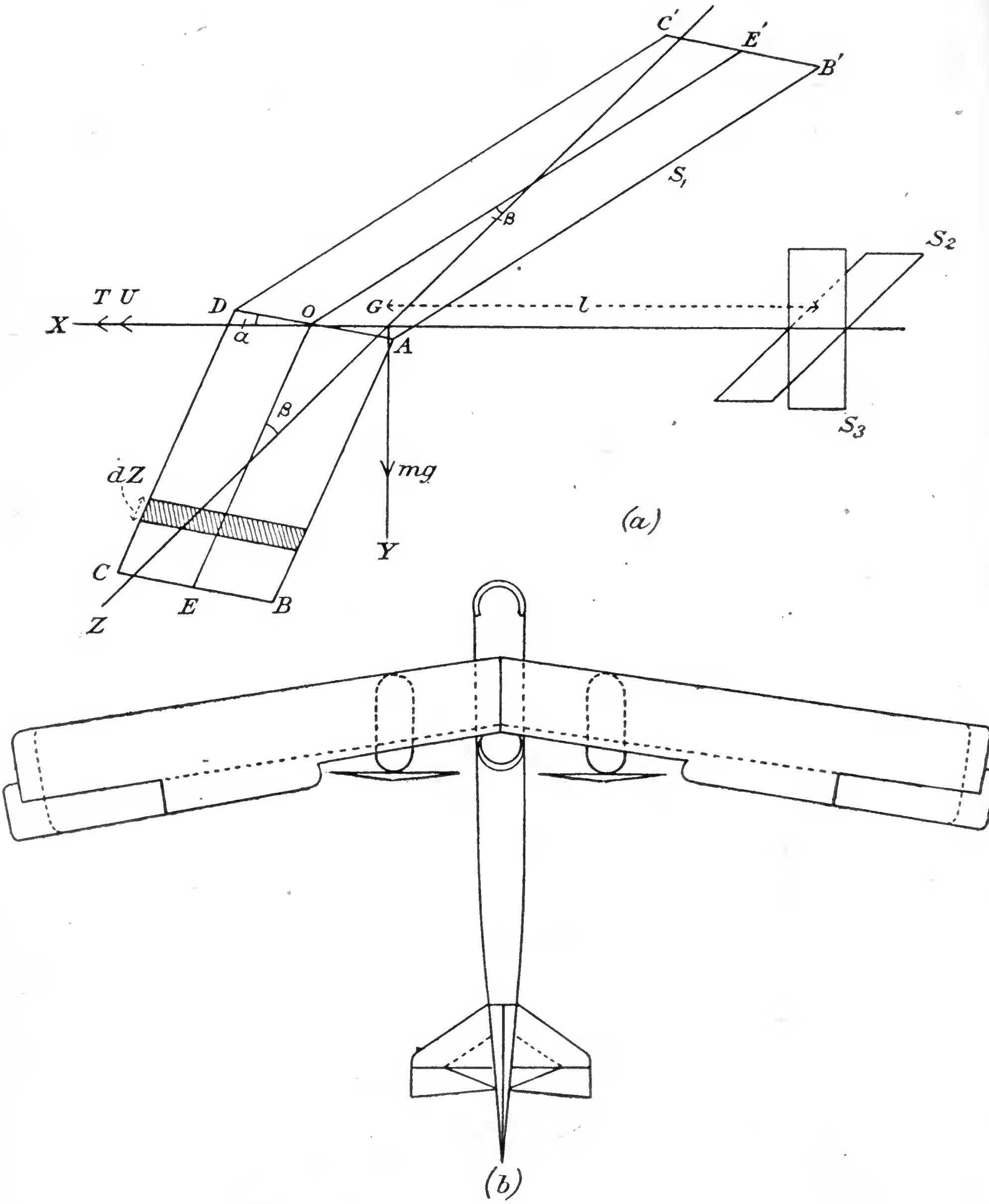

FIG. 113.—Lateral Stability: Swept-back Wings.

and we can approximate and write this condition in the form

$$n > \frac{4a}{3l} \tan a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Since a is usually small, it follows that a small value of n is sufficient to provide against E_2 being negative. Thus in practice $4a$ is about the same as $3l$, and then we get $n > \tan a$, which means that the dihedral angle must be (approximately) $< 180° - 2a$.

In reality, so large a value of the divergence from a flat plane is not required in aeroplanes, due to the presence of the body, the actual nature of air pressure, etc. We find, in fact, that too much "dihedral" introduces the danger of another form of instability, due to the possibility of C_2 and D_2 not being big enough to make the Routh's discriminant H_2 positive.

212. **Swept-back Wings.**—If we project the dihedral wings of Fig. 112 on to the plane of XZ, we get a slight angle between the projections of the two wings, suggesting an arrow. The swept-back wings are used in several important machines, and a brief indication of the effect of the arrow shape can be obtained in a manner similar to that adopted for the first simple type considered, and for the dihedral wings.

It is now necessary to put the wings a little in front of the centre of gravity of the machine, in order to give zero moment of air pressure in the steady motion, which is still assumed to be along the X (and propeller) axis, with neutral tail. A little consideration suffices to show that the mid-point O of the line of wing symmetry AD, and the mid-point of the line joining the mid-points of the tips, E, E', must be on opposite sides of and equidistant from G. If $EE' = 2a$, the span, then $GO = \frac{a}{2}\tan\beta$, where β is the angle between OE and the Z axis, the angle of the arrow-head EOE' being $180° - 2\beta$.

Let a be the angle of attack in the XY plane. If the equation of the plane wing $ABCD$ is

$$lX + mY + nZ = p,$$

then $Z = 0$ must give

$$Y = -\tan a\left(X - \frac{a}{2}\tan\beta\right),$$

and $Y = 0$ must give

$$Z = -\cot\beta\left(X - \frac{a}{2}\tan\beta\right).$$

Hence the equation of the plane $ABCD$ must be

$$m\tan a\left(X - \frac{a}{2}\tan\beta\right) + mY + m\tan a\tan\beta\,.\,Z = 0,$$

where

$$m^2(1 + \tan^2 a\sec^2\beta) = 1.$$

If we take an element of surface at distance Z from AD, defined by width dZ, the co-ordinates of its mid-point are

$$\left(-Z + \frac{a}{2}\right)\tan\beta,\ 0,\ Z, \qquad \qquad (66)$$

whilst its direction cosines are

$$m\tan a,\ m,\ m\tan a\tan\beta. \qquad \qquad (67)$$

Hence, putting $U_1 = U$, $U_2 = 0$, $U_3 = u_3$, $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$, $\Omega_3 = 0$ in (39), we get for the normal air pressure on this element

$$\frac{KS_1U^2m\tan a\,.\,dZ}{2a}\left\{1 + \frac{\tan\beta}{U}u_3 - \frac{Z}{U\tan a}\omega_1 + \frac{Z(1+\sec^2\beta) - \frac{a}{2}\tan^2\beta}{U}\omega_2\right\}. \qquad (68)$$

The components of force and couple due to this element of pressure are obtained by multiplying by the respective factors:

$$\left.\begin{array}{c} m\tan\alpha,\ \ m,\ \ m\tan\alpha\tan\beta, \\ -mZ,\ \ m\tan\alpha\left(Z\sec^2\beta-\frac{a}{2}\tan^2\beta\right),\ \ m\tan\beta\left(-Z+\frac{a}{2}\right). \end{array}\right\} \quad . \quad . \quad (69)$$

If we take the exactly symmetrical element of surface on the other wing $AB'C'D$, we use the same expressions for the normal pressure and the factors, provided we change β into $-\beta$, Z into $-Z$, and, where a occurs linearly with Z, a into $-a$. It is at once seen that the R_1, R_2 air forces and the G_3 couple are respectively

$$KS_1U^2m^2\tan^2\alpha,\ \ KS_1U^2m^2\tan\alpha,\ \ 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Thus the longitudinal forces and couples do not involve the lateral disturbances, and the moment about the Z axis is zero. The condition for steady motion is

$$KS_1U^2m^2\tan\alpha = \text{weight}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

The air force R_3, and the air couples G_1, G_2, due to the wings, are:

$$\left.\begin{array}{l} KS_1U^2m^2\tan^2\alpha\,.\tan\beta\left\{\dfrac{\tan\beta}{U}\,.\,u_3-\dfrac{a}{2U\tan\alpha}\,.\,\omega_1+\dfrac{a}{U}\,\omega_2\right\}, \\ -KS_1U^2m^2\tan\alpha\left\{\dfrac{a\tan\beta}{2U}\,.\,u_3-\dfrac{a^2}{3U\tan\alpha}\,.\,\omega_1+\dfrac{a^2}{12U}\,\omega_2\,(8+\tan^2\beta)\right\}, \\ KS_1U^2m^2\tan^2\alpha\left\{\dfrac{a\tan\beta}{2U}\,.\,u_3-\dfrac{a^2(4+\tan^2\beta)}{12U\tan\alpha}\,.\,\omega_1+\dfrac{a^2}{12U}(8+3\tan^2\beta+\tan^4\beta)\,\omega_2\right\}. \end{array}\right\} \quad (72)$$

As we are here mainly interested in E_2, we shall calculate only those derivatives required for E_2, namely:

$$c_1=-\frac{ga\tan\beta}{2k_1{}^2U^2},\quad e_1=-\frac{ga^2(8+\tan^2\beta)}{12k_1{}^2U^2},$$

$$c_2=\frac{S_3lg}{S_1U^2k_2{}^2m^2\tan\alpha}+\frac{ga\tan\alpha\,.\tan\beta}{2k_2{}^2U^2},\quad e_2=\frac{S_3l^2g}{S_1U^2k_2{}^2m^2\tan\alpha}+\frac{a^2g\tan\alpha}{12k_2{}^2U^2}(8+3\tan^2\beta+\tan^4\beta),$$

using the same notation as in § 208. We have taken account of the rudder in S_3. We find that E_2 is positive if, approximately,

$$\tan\beta>\frac{4a}{3l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

This is, of course, a surprisingly large angle. As in the case of the dihedral, we can, in practice, do with much less. But machines exist with β about 20°. It is, however, clear that the dihedral is the better way of securing freedom from spiral instability, since fancy shapes are a disadvantage both from the structural and from the aerodynamical point of view.

213. **Experimental Values of the Derivatives.**—Coming now to the experimental side of the problem of lateral stability in the case of the symmetrical aeroplane, we have to consider the values of

$$\begin{array}{lll} c_z, & d_z, & e_z, \\ c_1, & d_1, & e_1, \\ c_2, & d_2, & e_2. \end{array}$$

There is no difficulty about measuring the c derivatives; we simply experiment with relative wind velocities making small angles with the X axis, and lying in the XZ plane. In the case of the d and e derivatives we note that, in (52), e_z only occurs with U, and so can be neglected, whilst d_z can also be neglected, because it only occurs in C_2 and D_2, which are fairly big, and it is, at the same time, a small quantity. In the simple case of § 208 we found it to be zero. The d, e derivatives of G_1 and G_2 can be found by methods similar to that given for f_3.

For the first machine of which longitudinal particulars are given in § 206 the lateral derivatives are as follows:

$$\begin{array}{lll} Uc_z = 0\cdot25, & & \\ Uc_1 = -0\cdot033, & Ud_1 = 8, & Ue_1 = -2\cdot6, \\ Uc_2 = 0\cdot015, & Ud_2 = -0\cdot8, & Ue_2 = 1\cdot05. \end{array}$$

$k_1^2 = 25$, $k_2^2 = 35$. For horizontal flight the equation for λ is

$$(U\lambda)^4 + 9\cdot31\,(U\lambda)^3 + 9\cdot81\,(U\lambda)^2 + 10\cdot15\,(U\lambda) - 0\cdot161 = 0.$$

Hence there is spiral instability. We note that

$$H_2 \equiv B_2C_2D_2 - E_2B_2^2 - A_2D_2^2$$

is positive.

In a glide the equation for λ is found to be

$$(U\lambda)^4 + 9\cdot31\,(U\lambda)^3 + 9\cdot81\,(U\lambda)^2 + 10\cdot25\,(U\lambda) + 0\cdot467 = 0,$$

and H_2 is positive. There is, therefore, complete stability. We again have a case where the lower path is more stable.

In the case of the second machine in § 206 we have

$$\begin{array}{lll} Uc_z = 0\cdot108, & & \\ Uc_1 = -6\cdot028, & Ud_1 = 6\cdot68, & Ue_1 = -2\cdot16, \\ Uc_2 = 0\cdot012, & Ud_2 = -0\cdot67, & Ue_2 = 0\cdot861. \end{array}$$

and we find that the equation for λ in the case of horizontal flight is

$$(U\lambda)^4 + 7\cdot53\,(U\lambda)^3 + 6\cdot2\,(U\lambda)^2 + 7\cdot38\,(U\lambda) - 0\cdot0756 = 0.$$

We again have spiral instability. H_2 is positive.

214. **The Lateral Oscillation.**—This information about actual machines will enable us to discuss the biquadratic for λ in the problem of lateral stability in a manner similar to that adopted in the case of longitudinal stability. The theory is again due to Bairstow.

We note that if we write the biquadratic in the form

$$A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

then E_2 is always very small compared with D_2. In most cases D_2 is positive, whilst E_2 may be slightly greater or slightly less than zero. The fact that E_2 is small is also evident from Bryan's method of the idealised aeroplane. It is, therefore, clear that $\lambda_2 = - E_2/D_2$ is very nearly a solution, and if we refer to the numerical cases in the last article, we find that the values of λ thus obtained are so small that the approximation is really a very good one.

Again, since we find that B_2 is a large quantity, about 10 times A_2/U, the large value of λ obtained from

$$A_2\lambda + B_2 = 0 \quad \ldots \ldots \ldots \ldots \quad (75)$$

is also an approximation. To find the correction we put $\lambda = - B_2/A_2$ in the right-hand side of

$$A_2\lambda + B_2 = -\frac{C_2}{\lambda} - \frac{D_2}{\lambda^2} - \frac{E_2}{\lambda^3}$$

E_2 is very small, C_2 and D_2 are quantities of the same order as B_2. Hence a more exact value of λ is obtained from

$$A_2\lambda + B_2 = - C_2\Big/\left(-\frac{B_2}{A_2}\right) = \frac{A_2C_2}{B_2},$$

i.e.

$$A_2\lambda + \frac{B_2{}^2 - A_2C_2}{B_2} = 0. \quad \ldots \ldots \ldots \ldots \quad (76)$$

Taking now the factors

$$\left(\lambda + \frac{E_2}{D_2}\right)\left(A_2\lambda + \frac{B_2{}^2 - A_2C_2}{B_2}\right),$$

we write

$$A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2 \equiv$$
$$\left(\lambda + \frac{E_2}{D_2}\right)\left(A_2\lambda + \frac{B_2{}^2 - A_2C_2}{B_2}\right)\left(\lambda^2 + F_2\lambda + \frac{B_2D_2}{B_2{}^2 - A_2C_2}\right) = 0, \quad \ldots \quad (77)$$

where F_2 is to be chosen so as to give a good approximation. If we compare coefficients in the two assumed identical expressions, we get for the coefficients of λ^3 the relation

$$\frac{B_2}{A_2} = \frac{B_2{}^2 - A_2C_2}{A_2B_2} + \frac{E_2}{D_2} + F_2,$$

which gives

$$F_2 = \frac{C_2}{B_2} - \frac{E_2}{D_2} = \frac{C_2D_2 - E_2B_2}{B_2D_2}.$$

We, therefore, use the factors

$$\left(\lambda + \frac{E_2}{D_2}\right)\left(\lambda + \frac{B_2{}^2 - A_2C_2}{A_2B_2}\right)\left(\lambda^2 + \frac{C_2D_2 - E_2B_2}{B_2D_2}\lambda + \frac{B_2D_2}{B_2{}^2 - A_2C_2}\right) = 0, \quad \ldots \quad (78)$$

if certain limitations are satisfied. These limitations are supplied by the coefficients of λ^2 and λ, since the coefficients of λ^4, λ^3, and the constant term have been made to agree.

The second coefficient, $\dfrac{C_2D_2 - E_2B_2}{B_2D_2}$, at once suggests, by analogy with the longitudinal case, an approximate form of the H_2 condition of lateral stability. Now in practice, as already seen, if A_2 is unity, then B_2, C_2, D_2 are quantities of about the order 10, divided respectively by U, U^2, U^3, whilst E_2 is very small. It is thus obvious that, in practice, $B_2{}^2 - A_2C_2$ and $C_2D_2 - E_2B_2$ are both amply positive. We can, therefore, approximate, without any fear of thereby hiding a case of instability, by writing the factors in the simplified form

$$\left(\lambda + \frac{E_2}{D_2}\right)\left(\lambda + \frac{B_2}{A_2}\right)\left(\lambda^2 + \frac{C_2}{B_2}\lambda + \frac{D_2}{B_2}\right) = 0. \quad \ldots \ldots \quad (79)$$

We have the important result that, in practice, instability can arise from the first factor. The other factors (which also include Routh's discriminant) only represent disturbances that die away.

The factor $\lambda + \frac{B_2}{A_2} = 0$ represents a disturbance which is not an oscillation, but one that merely diminishes continually to zero. The rate of decay is very fast, the corresponding solution being e^{-UB_2t/A_2}, which is about e^{-10t}, so that the disturbance is reduced to one-half in a small fraction of a second.

The factor $\lambda^2 + \frac{C_2}{B_2}\lambda + \frac{D_2}{B_2} = 0$, which is roughly

$$(U\lambda)^2 + (U\lambda) + 1 = 0,$$

represents a damped oscillation of period about $4\pi/\sqrt{3}$, *i.e.* 7 seconds; the amplitude is reduced to one-half in about $1\frac{1}{2}$ seconds. The nature of the motion has been described as the ***Dutch roll*** (Hunsaker). It can be shown that D_2 can be made negative by means of a very sharp dihedral angle. In this case the Dutch roll can lead to instability. The machine yaws (or turns) to the right and to the left, at the same time rolling and side-slipping.

The first factor, $\lambda + \frac{E_2}{D_2} = 0$, is the one that can give rise to lateral instability. In an ordinary machine D_2 is positive. The coefficient E_2, as already pointed out, depends on the dihedral angle. If this is sufficient there is stability, the disturbance merely dying out without oscillation. But if the dihedral angle is insufficiently sharp, E_2 can be negative, and then there is spiral instability, resulting in a spiral nose dive. This should, if possible, be prevented by proper adjustment of the dihedral and vertical fin.

215. **The Propeller; Efficiency, Pitch, Slip.**—We have so far neglected changes in the air forces due to the propeller. The theory of this branch of aeroplane mechanics is somewhat difficult, and we shall here give only a brief account of the manner in which the propeller is made to produce a thrust by means of its revolutions, forced on it through the agency of an engine. The method will be the **blade element theory,** which is, in its essence, the same as that used by Bryan for the lateral stability of an idealised aeroplane.

Before we attack the problem in detail, let us see what information we can obtain from the general theory of fluid pressures. Referring to Chapter I., § 34, *et seq.*, let us consider the air pressure on a body whose motion consists of a velocity U along the X axis, and an angular velocity Ω about the X axis, *i.e.* where motion is defined by a screw motion. Omitting the elasticity of the medium, and neglecting the time element, we have, from equation (26), Chapter I.,

$$R_1 = \rho l^2 U^2 \chi_x\left(\frac{\nu}{lU}, \frac{l\Omega}{U}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

with corresponding expressions for R_2, R_3, G_1, G_2, G_3.

We shall show that we may leave out the argument ν/lU from the

expressions for the force and couple components. To find if this is possible, we argue as follows. Suppose that

$$R_1 = \rho l^2 U^2 \chi_x \left(\frac{l\Omega}{U}\right), \text{ etc}$$

Then $R_1 = 0$ when $\chi_x(l\Omega/U) = 0$. This means that we get no thrust along the X axis for a certain definite ratio Ω/U. Is this so? Experiment shows that such is the case. The thrust of a propeller is zero for a definite value of Ω/U, no matter how the actual individual values of Ω and U may vary. As the viscosity does not enter into R_1, there is no reason why it should affect the other components of force (these are actually zero in a symmetrical propeller) and the components of couple. Using the fact that in a propeller, with flight along the axis of the propeller, $R_2 = R_3 = G_2 = G_3 = 0$, we write

$$\left.\begin{aligned} R_1 &= \rho l^2 U^2 \chi_x \left(\frac{l\Omega}{U}\right), \\ G_1 &= \rho l^3 U^2 \chi_1 \left(\frac{l\Omega}{U}\right). \end{aligned}\right\} \qquad (81)$$

The **efficiency** of this body, considered as a propeller, is the ratio of the work done in a given time by the R_1 force to the work that has to be done by the engine in the same time to overcome the couple G_1. Hence the efficiency is

$$\eta = \frac{UR_1}{\Omega G_1} = \frac{U}{l\Omega} \frac{\chi_x\left(\frac{l\Omega}{U}\right)}{\chi_1\left(\frac{l\Omega}{U}\right)}, \qquad (82)$$

so that the efficiency is a function of $l\Omega/U$. For a given propeller it is a function of the ratio Ω/U.

We now introduce the **pitch** of the propeller, which is defined as *the distance it moves forward whilst it makes one complete revolution.* The time for a revolution is $2\pi/\Omega$, Ω being measured in radians per second. Hence the pitch

$$p = \frac{2\pi U}{\Omega} \text{ units of length.} \qquad (83)$$

It follows that the efficiency for a given propeller is a function of the pitch.

In designing a propeller, our object should be to get the greatest possible efficiency for the pitch in use in the ordinary flight of the machine. In practice, it is customary to define another quantity, the **slip.**

Let p_0 be the value of the pitch for the ratio Ω/U, giving zero thrust. Then the slip is defined as

$$s = \frac{p_0 - p}{p_0} = 1 - \frac{p}{p_0}. \qquad (84)$$

It is found that the smaller p/p_0, the greater is the thrust produced by the propeller. Hence the thrust increases with the slip.

We now proceed to the consideration of the idealised propeller.

216. **The Blade Element Theory; Zero Thrust.**—In Fig. 114 let the axes G, X, Y, Z be the body axes already defined. Let O' be the centre of the propeller, and consider an element of area $ABCD$ whose width AB is some function of its distance from O', *i.e.* of Z, and length AD is dZ. The co-ordinates of its mid-point can be taken to be $(0, 0, Z)$. Imagine this element of area to be obtained from the rectangle $A'B'C'D'$ in the plane through O' parallel to the YZ plane by means of a rotation through an angle a, which we shall take to be dependent on the value of Z; a is to be positive when obtained by a rotation from X to Y.

If this element is made to rotate about the line $O'X$ from Y to Z, we get an air pressure on the negative side, *i.e.* in the figure on the same side as the origin G. But G is not fixed. The whole set of axes is moving in the direction GO' with a certain velocity. We readily see that if this

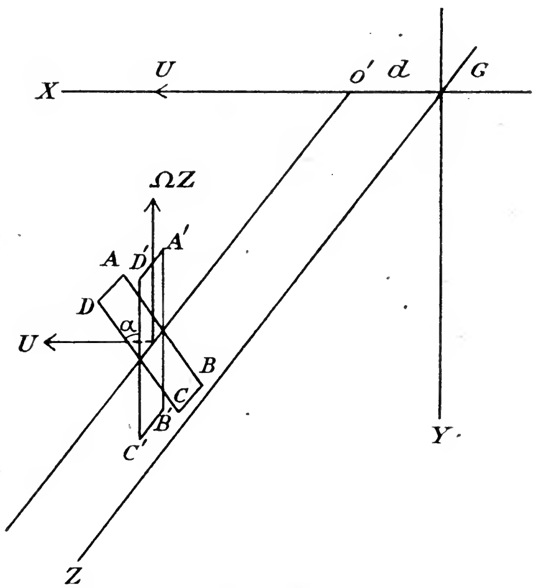

Fig. 114.

velocity is small we get a thrust on the G side, but that if the velocity is very large we get a thrust on the opposite side, which gives a force from X to G. There is a certain relation between the velocity of translation and the velocity of rotation that gives just no thrust on the element. To obtain this relation, we have to put down the condition that the motion relatively to the air (assumed at rest) shall be tangential to the element.

Let U_0 be the velocity of translation, and Ω_0 the angular velocity. Then the element considered has velocity components U_0 parallel to GO', and $\Omega_0 Z$ parallel to YG. Hence we get no thrust if $\Omega_0 Z \sin a = U_0 \cos a$. But if X, Y, Z are the co-ordinates of a point in the element, we have $\frac{(X-d)}{Y} = -\tan a$, where d is the length GO'. Hence we get no thrust if the element satisfies the equation

$$\frac{X-d}{Y} = -\frac{U_0}{\Omega_0 Z}. \qquad (85)$$

Using the equation (83), the equation for the element at Z is, therefore, for zero thrust,

$$(X-d)Z+\frac{p_0}{2\pi}Y=0. \qquad (86)$$

It is easily seen that we get the same condition for an element at distance Z on the other side of O'. Thus the two-bladed propeller will produce no thrust if its form is tangential to the parabolic hyperboloid

$$(X-d)Z+\frac{p_0}{2\pi}Y=0.$$

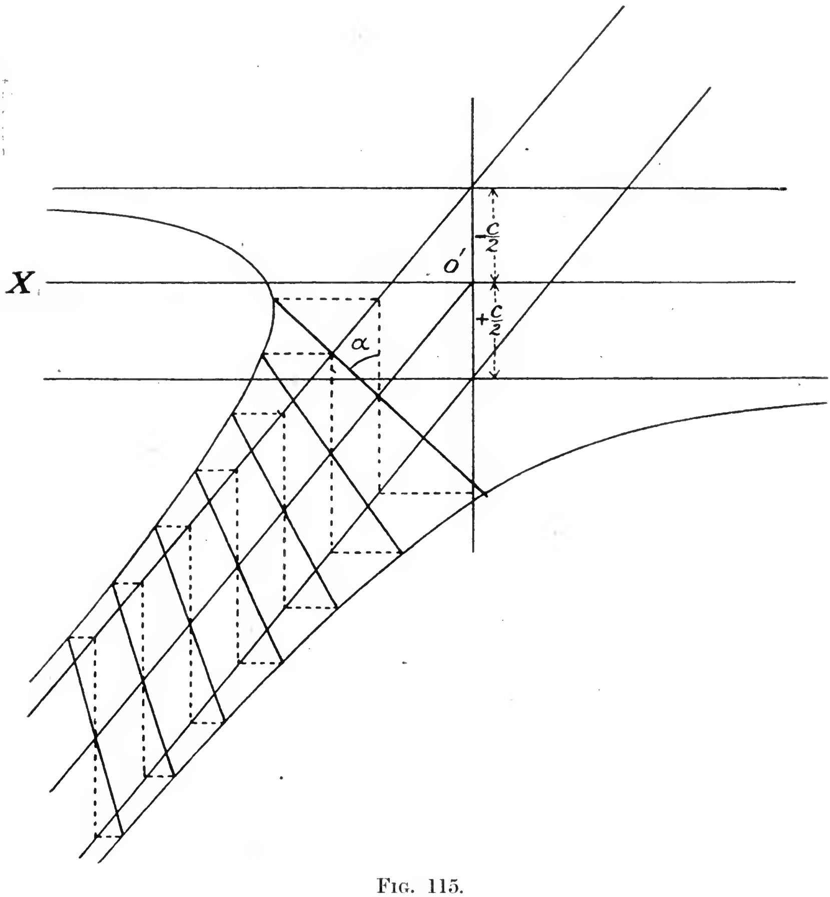

FIG. 115.

The form of this no-thrust propeller is easily plotted. Take Y to have two values $\pm\frac{C}{2}$, and draw the rectangular hyperbolas in the planes $Y=\frac{C}{2}$, $Y=-\frac{C}{2}$. The direction of any element dZ at Z is obtained by joining the two points on these two hyperbolas, which have the given value Z (Fig. 115).

217. **Thrust-producing Propeller.**—We have now to convert this no-thrust screw into a propeller giving a thrust. Let us take once

more the element $ABCD$ in Fig. 114. If a has been chosen to satisfy condition (86) for zero thrust on this element, we can either make the propeller have a smaller pitch p than the one p_0 for which the screw has been constructed, *i.e.* we can keep the screw as it was, but increase the slip from zero to some suitable value by choosing the appropriate ratio Ω/U. Or we can bend the element round through a further angle, say ϕ, so that it now makes an angle $(a + \phi)$ with the plane through O' parallel to YZ.

The first method would have the advantage of giving a very easy formula for the efficiency of the propeller. As already seen in § 215, since the propeller remains the same, the efficiency depends only on the pitch. We can show that it would be actually p/p_0.

For if dR is the normal pressure on each of the two elements at distance Z from the axis, the corresponding R_1, G_1 contributions are

$$2dR \,.\, \cos a \quad \text{and} \quad 2dR \,.\, Z \sin a.$$

The efficiency of this pair of elements, *i.e.* the work done by the thrust they produce compared with the work necessary to push the elements through the air, is

$$\frac{2dR \,.\, \cos a \,.\, U}{2dR \,.\, Z \sin a \,.\, \Omega} = \frac{U}{\Omega Z \tan a}.$$

But for zero thrust

$$\tan a = \frac{U_0}{\Omega_0 Z}.$$

Hence the efficiency is

$$\frac{U\Omega_0}{U_0\Omega} = \frac{p}{p_0},$$

the same for each such pair of elements, and, therefore, for the whole propeller.

But in practice this method is not adopted. We proceed to the consideration of the second process.

218. **Aerofoil Elements.**—In this process we start off with the no-thrust screw for the given velocity of the machine and angular velocity of the propeller. If a has been determined for any value of Z, we increase the inclination of the element to $(a + \phi)$. If dR is the normal pressure on each of the two elements at distance Z from the axis, it contributes

$$2dR \,.\, \cos (a + \phi), \quad 2dR \,.\, Z \sin (a + \phi)$$

to R_1 and G_1 respectively. The efficiency of the element is, therefore,

$$\frac{U_0}{\Omega_0 Z_0 \tan (a + \phi)} = \frac{\tan a}{\tan (a + \phi)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (87)$$

The pitch is no longer constant for all the elements, unless ϕ is chosen for each value of a, so that $\tan a/\tan (a + \phi)$ is constant—and this reduces to the method already considered. We need some criterion to define ϕ.

This criterion is supplied by the nature of the functions and general shape of the propeller. Since there are huge pressures on each blade, there is a great bending moment at any section of the blade, and this moment increases as we approach the centre or boss. Thus the blade must

not only be fairly thick, but the thickness must increase as we approach the boss. We are no longer dealing with a plane lamina for which the normal pressure is given by the sine of the angle of attack. Instead, we have for each element an aerofoil of considerable thickness, and we, therefore, take advantage of this fact to give the element a camber, the extent of which is determined by the thickness it is thought desirable to assign to the element and the width of the blade at the element.

Now experiment has shown that for a given shape of aerofoil section there is an angle of attack producing the greatest possible efficiency for the corresponding element of a propeller. Let L, D be the components of air pressure produced on the element, referred to the direction of the element in the no-thrust screw, Fig. 114, *i.e.* L is the lift or thrust normal to the direction in which the air appears to approach the element, D is the drag or thrust along this direction. Then the component of thrust parallel to U is

$$L \cos a - D \sin a,$$

and the component which has to be overcome by the engine is

$$L \sin a + D \cos a.$$

The useful work done is, therefore, for the pair of elements

$$2U_0 (L \cos a - D \sin a),$$

and the work that must be done by the engine is

$$2\Omega_0 Z (L \sin a + D \cos a).$$

The efficiency is, therefore,

$$\eta = \frac{U_0}{\Omega_0 Z} \frac{L \cos a - D \sin a}{L \sin a + D \cos a}.$$

Putting $L = D \cot \phi$, we get

$$\eta = \frac{U_0}{\Omega_0 Z} \cot (a + \phi),$$

i.e.

$$\eta = \frac{\tan a}{\tan (a + \phi)},$$

which is equation (87) with ϕ as here defined.

We, therefore, conclude that to get the best efficiency out of the element we must make ϕ a minimum for the given aerofoil section, which means that we must choose the direction of the element such that the relative wind produces the maximum ratio L/D.

In practice this means that the elements are turned from the no-thrust screw more and more as we approach the boss, because, in general, the greater the camber of an aerofoil, the greater is the angle between the relative wind and the chord of the aerofoil for maximum ratio L/D.

The effect is that we can no longer apply easy mathematical methods to the discussion of the propeller. The methods used are graphical, and the subject is outside the scope of this work. For a detailed account of the theory and methods the student should refer to the special books on the subject, notably that by Riach.

Account must also be taken of the fact that the air, when it is approached by an element, is not really in a state of rest, but is sucked inwards in what is known as the **slip stream.** This has the effect of increasing the apparent value of U_0 and, therefore, of the pitch p_0. The value of a is correspondingly increased, and each element has to be turned an additional small angle in order to obtain maximum efficiency.

219. **Resistance Derivatives.**—In our investigations on the stability of the aeroplane we have supposed the resistance derivatives to include parts due to the propeller. We can now form an idea of how this is done. If the steady motion is along the axis of the propeller, then it is found that the only resistance derivative that is of any importance is the change in the thrust due to changed X velocity. If the angular velocity of the propeller is kept constant, this means changed pitch. Hence a positive value of u_1 (Chapter III., § 87), which is equivalent to an increased value of U, gives increased pitch, and, therefore, reduced thrust. Thus the derivative a_x used in § 86 must be decreased by an amount representing the derivative due to the propeller.

EXERCISES

1. Assuming that the sail of a yacht experiences a wind pressure following the sine law, find how to obtain the maximum speed in a given direction, with the wind blowing in some given different direction.

2. A plate falls vertically with its plane horizontal; investigate the motion and prove that it is unstable.

3. A plate attached to a heavy body, so that the centre of gravity is below the plate, falls vertically as a parachute. Shew that for stability the centre of gravity must lie within certain limits.

In Exercises 2 and 3, introduce the shift of the centre of pressure and the rotary effects by means of the following considerations: If U is the resultant velocity and a is the angle of attack, whilst the angular velocity is ω, then the normal air pressure at the centre of the plate and the air-moment about the centre of the plate can be written

$$R = KSU^2 \sum_0^\infty A_n \sin(2n+1)a + 2KSU\omega \sum_0^\infty A_n' \cos(2n+1)a;$$

$$G = KSU^2 \sum_1^\infty E_n \sin 2na + 2KSU\omega \sum_0^\infty E_n' \sin 2na + KSC\omega^2;$$

where the A's, A''s, E's, E''s, C are constants independent of the motion. It is a useful exercise to justify these expressions from general considerations of symmetry, etc.

4. The body in Exercise 3 is held loosely at a point in the line through the centre of the plate perpendicular to it, in a given horizontal wind. Find the position of equilibrium.

5. A kite consists of a plane lamina with a string attached to its mid point, which is also the centre of gravity. It is held in a given wind. Find the position of equilibrium.

6. If the string in Exercise 5 is attached to any other point of the plane lamina, investigate the equilibrium.

7. A rigid body whose moment of inertia is negligible (as in Lanchester's phugoids) moves in a resisting medium, and no other forces act on it. Prove that the path of the centre of gravity is a circle.

8. Prove that a plane lamina moving in its own plane in a resisting medium is unstable.

9. In an idealised aeroplane assume that the body, etc., produces a head resistance proportional to the square of the velocity. Then, using the notation of § 191, we have $mg = KS_1U^2 \sin a \cos a$, $T = KS_1U^2 \sin^2 a + K'U^2$ for normal motion. Plot the necessary horse-power as a function of U, and deduce that the speed must lie between certain limits if the engine is sufficiently powerful to make steady motion possible. (Assume a small.)

10. Find the least horse-power necessary, and show that the angle of attack is then $\sin^{-1} \sqrt{3K'/KS_1}$.

11. If the aeroplane in Exercise 9 moves at an angle θ below the horizontal, with velocity U along the propeller axis (neutral elevator), write down the equations for steady motion, and find the necessary horse-power, assuming a and θ to be small. In general, show how the graph of Exercise 9 can be used to determine how fast an aeroplane can climb (Performance Curve).

12. Assuming that air pressures vary as the density of the air, shew that with given aeroplane and engine there is a level above which steady flight is impossible (Ceiling). Find the ceiling for normal flight, and deduce from Exercise 9 the ceiling for any kind of horizontal flight, with given available horse-power.

13. Examine the effect of loading on rectilinear flight. Shew how to find the greatest load for given horse-power.

14. Defining normal circling flight as being that in which the side-slip is zero and the propeller axis horizontal, show that the angle of bank required for radius r is $\sin^{-1}(U^2/gr)$, where U is the velocity in normal rectilinear flight. Find the smallest radius for given horse-power.

15. Show that in circling flight the necessary lift is the same as if the weight had been increased in the ratio of the secant of the angle of bank.

16. Show that in changing from rectilinear to circling flight, keeping the power constant, the angle of attack must be increased.

CHAPTER VIII

AEROPLANE IN MOVING AIR

220. **Steady Wind.**—In considering the motion and stability of an aeroplane, we have assumed the air to be at rest, by which we have meant that the only motions that exist in the air are due exclusively to the motion through it of the aeroplane. It is now necessary to remove this restriction, since in practice the air may, and generally does, possess motion of its own, either in the nature of a steady wind or in the nature of a gust. We take first the case when there is a steady wind.

221. **In the Plane of Symmetry.**—If the wind is horizontal and its direction parallel to the vertical plane of symmetry, let its velocity be U_1' parallel to the x axis. In the statical equilibrium for steady horizontal motion, the air pressure is due to the motion of the aeroplane relatively to the air. The effect is, therefore, *aerodynamically* as if the

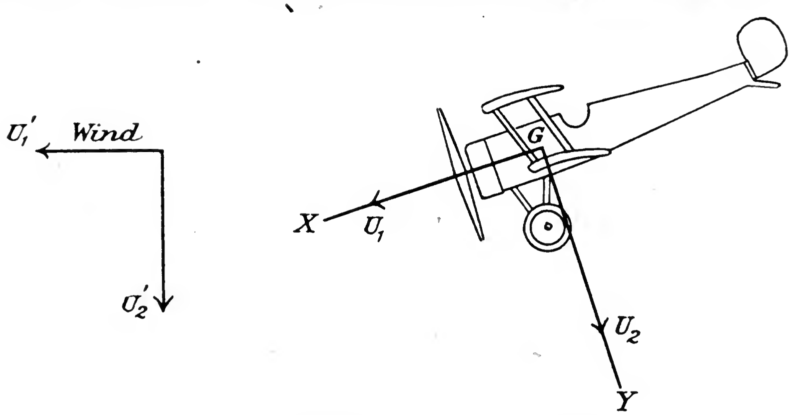

Fig. 116.—Wind in Plane of Symmetry.

aeroplane moved with velocity $U_1 - U_1'$ along the X axis, instead of U_1, the actual X velocity. This can be expressed by saying that the aeroplane is carried by the wind with velocity U_1', whilst in its motion relatively to the wind we use the investigations of Chapter III. with $U_1 - U_1'$ substituted for U_1.

This result evidently applies to the case of the wind inclined to the horizontal. Taking the problem in its general form, let the wind, Fig. 116, have velocity components U_1', U_2' parallel to the x, y axes which are fixed in space. If the aeroplane itself has velocity components U_1, U_2

along the X, Y axes, then we say that the aeroplane moves with the given wind, whilst relatively to the wind it moves with velocity components

$$U_1 - U_1' \cos\Theta - U_2' \sin\Theta,\quad U_2 + U_1' \sin\Theta - U_2' \cos\Theta,$$

where Θ is the angle between the X, Y axes and the x, y axes. These components must be used instead of U_1, U_2 in the more general investigations of Chapter III. Otherwise the problems of stability, etc., remain exactly the same as before.

222. **Steady Motion.**—It is thus clear that the steady longitudinal wind can be taken into account with very little trouble. Consider the problem of steady motion.

In horizontal flight with propeller axis horizontal, *i.e.* normal flight, and with horizontal wind, the equilibrium conditions are

$$T_0 = R_1(U_1 - U_1'),\quad g = R_2(U_1 - U_1') \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

the condition of moments being unaffected. R_1, R_2 are functions of $U_1 - U_1'$, generally proportional to the square of this relative velocity, but we need not assume this at present. The fact that $R_2 = g$ determines $U_1 - U_1'$, and since U_1' is supposed known, the velocity U_1 is determined. The propeller thrust is also found.

In the general case the conditions of steady flight are :—

$$\left.\begin{aligned} T_0 + g\sin\Theta_0 &= R_1(U_{10} - U_1'\cos\Theta_0 - U_2'\sin\Theta_0,\ U_{20} + U_1'\sin\Theta_0 - U_2'\cos\Theta_0), \\ g\cos\Theta_0 &= R_2(U_{10} - U_1'\cos\Theta_0 - U_2'\sin\Theta_0,\ U_{20} + U_1'\sin\Theta_0 - U_2'\cos\Theta_0), \\ 0 &= G_3(U_{10} - U_1'\cos\Theta_0 - U_2'\sin\Theta_0,\ U_{20} + U_1'\sin\Theta_0 - U_2'\cos\Theta_0). \end{aligned}\right\} (89)$$

Since $G_3 = 0$, we get for a given machine the direction of motion relatively to the machine, *i.e.* the ratio of the velocity components

$$U_{10} - U_1'\cos\Theta_0 - U_2'\sin\Theta_0,\quad U_{20} + U_1'\sin\Theta_0 - U_2'\cos\Theta_0.$$

If the propeller thrust is given, the first two equations of (89) determine Θ_0 and the actual values of the relative velocity components. Since U_1', U_2' are supposed known, we deduce U_{10}, U_{20}, in the actual motion relatively to the machine and, therefore, the actual motion in space. Hence for a given machine and given propeller thrust the steady motion is completely determined in a given steady wind.

Similarly, in a given steady wind, in order to get a given velocity in space, it is necessary to get a given velocity relatively to the wind. $G_3 = 0$ determined the ratio of the X, Y components of this relative velocity. Hence their components are determined. R_1, R_2 are, therefore, known, so that equations (89) give Θ_0 and T_0. Thus for a given machine and given steady wind, a definite propeller thrust is required for a prescribed steady velocity, and the direction of this velocity is determined.

To get a definite velocity in a definite direction, the machine itself must be adjusted to these requirements. This is done by means of the engine and elevator as already explained in Chapter III.

223. **Horizontal Side Wind.**—If the horizontal wind is not in the plane of symmetry whilst the aeroplane is symmetrical and flying horizontally with propeller axis horizontal, let U_1', U_3' be the X, Z velocity components of the wind. Then the aeroplane will move with the component U_3', and along the propeller axis the problem is the same as

before. It follows that as seen from the ground the aeroplane will appear to fly sideways at an angle $\tan^{-1}(U_3'/U_1)$ to its axis, Fig. 117.

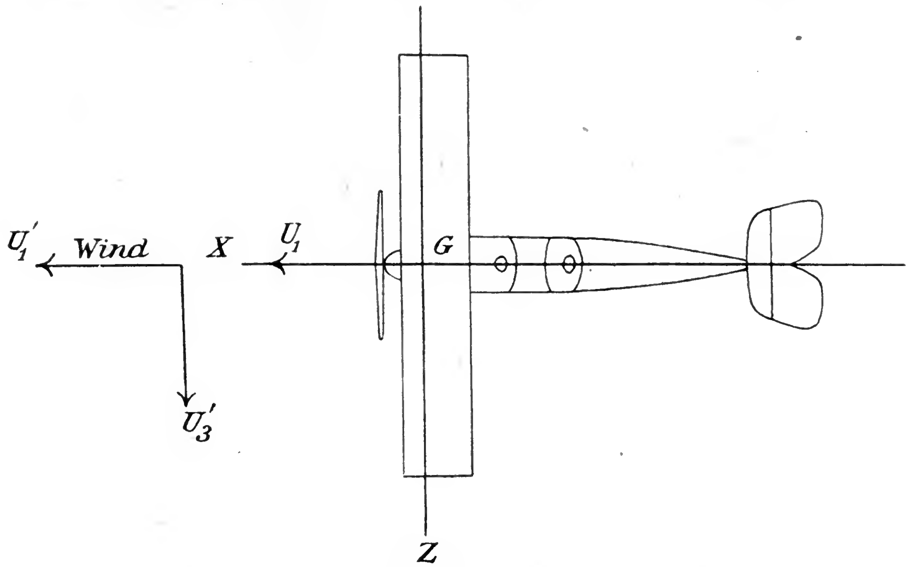

FIG. 117.—Horizontal Wind not in Plane of Symmetry.

The more practical problem is, however, when the motion of the air is known and we wish to find out how the aeroplane will fly. Let U_1'

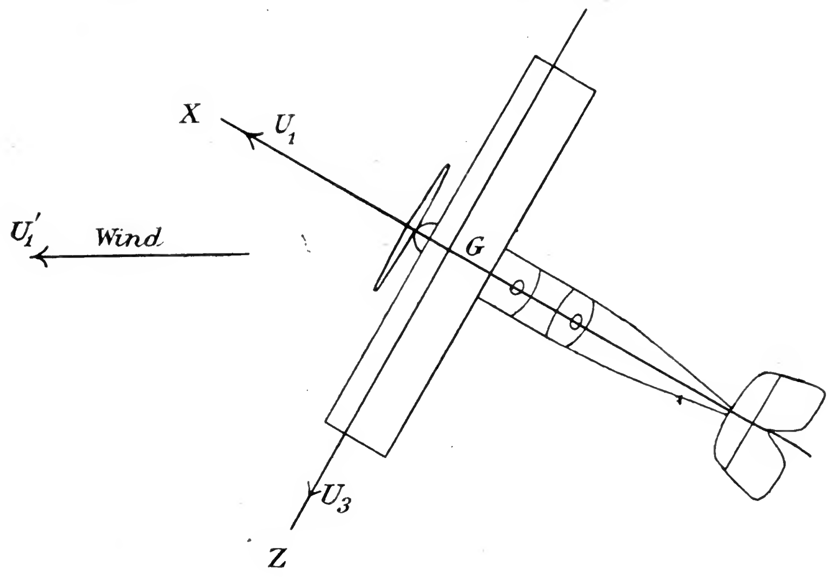

FIG. 118.—Horizontal Wind not in Plane of Symmetry: Direction of Wind taken as x Axis in Space.

(Fig. 118) be the velocity of the wind, assumed horizontal, and take this for the direction of the x axis in space. If U_{10}, U_{30} are the X, Z velocity

components of the aeroplane, and Ψ_0 the angle that its axis makes with the direction of the wind in the sense determined by the conventions of § 106, then since the aeroplane must fly symmetrically relatively to the wind, we get

$$U_{30} - U_1' \sin \Psi_0 = 0.$$

Also, the relative motion of the aeroplane is to be taken with X velocity:

$$U_{10} - U_1' \cos \Psi_0.$$

The latter velocity is determined by the shape and weight of the machine; U_1' is supposed known. Hence if Ψ_0 is given, U_{10} and U_{30} are determined, *i.e.* the motion of the aeroplane as seen from the earth's surface. Also the required propeller thrust is known.

224. **General Case.**—When the wind has components both vertically and horizontally out of the plane of symmetry of the machine, the motion can be discussed in a similar manner. But as we must now introduce the three-dimensional methods of Chapter IV., we shall at once take the most general problem of the steady motion of any aeroplane in any steady wind.

We take the x axis to be parallel to the horizontal component of the wind's velocity, so that Ψ_0 of § 106 is measured from the vertical plane containing the wind's motion. Θ_0, Φ_0 are as there defined. Let U_1', U_2' be the horizontal and vertical components of the wind's velocity along the axes of x, y. If U_{10}, U_{20}, U_{30} are the X, Y, Z velocity components of the aeroplane in steady motion, then the X, Y, Z components relatively to the wind are, by Figs. 55–7:—

$$\left.\begin{aligned}
&U_{10} - U_1' \cos \Psi_0 \cos \Theta_0 - U_2' \sin \Theta_0, \\
&U_{20} - U_1'(-\cos \Psi_0 \sin \Theta_0 \cos \Phi_0 + \sin \Psi_0 \sin \Phi_0) - U_2' \cos \Theta_0 \cos \Phi_0, \\
&U_{30} - U_1'(\ \ \cos \Psi_0 \sin \Theta_0 \sin \Phi_0 + \sin \Psi_0 \cos \Phi_0) + U_2' \cos \Theta_0 \sin \Phi_0.
\end{aligned}\right\} \quad . \quad . \quad (90)$$

Hence in the equations of statical equilibrium for rectilinear steady motion Chapter IV. (138), we use the expressions (90) instead of U_{10}, U_{20}, U_{30} in the air forces and couples as well as in the propeller thrust. The conditions $G_{10} = G_{20} = G_{30} = 0$ give us the ratios of the relative velocity components, since for a given machine these conditions determine the direction relatively to the machine of the motion relatively to the wind. We now have six equations to find the six quantities U_{10}, U_{20}, U_{30}, Θ_0, Φ_0, T_0. The angle Ψ_0 must be given, of course, since there is nothing to determine the azimuths of aeroplane velocities (§ 116).

In the same way we can find the steady circular motion and steady spiral motion relatively to any given wind for given shape, etc., of the aeroplane.

The effect of the wind is made use of in practical flight to assist in climbing as well as to reduce the speed relatively to the earth in landing. In both cases it is clearly advantageous to set the nose of the machine pointing against any horizontal wind there may be blowing at the time. Again if the wind happens to be blowing in the direction in which it is desired to fly, the effect is to increase the actual speed of flight by the speed of the wind. If, however, there is a contrary wind, the effect is obviously detrimental to flight. The study of the winds is, therefore, of fundamental importance in practical aeronautics, and the science of meteorology must,

therefore, play an important part in the development of commercial and general aeronautics. A knowledge of the currents in the air will often enable a pilot to increase his effective speed very considerably by choosing a level above the earth's surface at which the direction of the wind is that in which he desires to fly.

225. **Wind Changes.**—While there is no difficulty in discovering the steady motion of an aeroplane when there is a known steady wind blowing, the problem becomes very difficult when we wish to discuss the effect of the wind in general. To show that this must be the case let us enquire into the effect of a sudden change in the state of the air. Suppose, *e.g.*, that an aeroplane is flying steadily in a straight line in still air, and then there is suddenly a change in the air, the change being represented by a given steady wind. We have seen that with a steady wind the motion of the aeroplane is *relatively* the same as if there were no wind. Thus the appropriate steady motion in space is now obtained by taking the relative steady motion of the aeroplane and adding on the velocity of the wind. But at the instant when the change in the wind takes place the motion of the aeroplane is not yet changed because of its inertia. Thus the actual motion differs from the appropriate steady motion and we have the problem of the general motion of the aeroplane.

We can attack this problem in one of two extreme ways. We can either consider the motion of the aeroplane referred to axes fixed in space, and look upon the wind as causing a disturbance in this motion; or, we can consider the motion of the aeroplane referred to axes *moving with the air*, and look upon the initial steady motion as a disturbance from the final steady motion. Both methods possess advantages in practice. The first is clearly preferable when the pilot's aim is to get to a definite place on the earth's surface, as *e.g.* if he wants to alight. The second is preferable when he wishes to continue flying, so that he is desirous of establishing steady motion with the wind effect included.

Mathematically, it is slightly simpler to use the first method, as we already have moving axes in the body, and we should avoid the further complication of moving axes in space. When we have discovered the motion in space, it is an easy matter to find the motion relatively to the air by subtracting the velocity of the wind. We shall, therefore, consider the motion referred to the earth's surface.

The aeroplane's steady motion with the wind included can be found as already explained. But it has a different motion at the instant when the wind arises. We have, therefore, the problem suggested above, § 72, and we can use the method of initial motions to discuss the effect of the wind, and the method of "oscillations" to discuss the stability, *i.e.* whether the aeroplane will tend to assume the new steady motion. But the stability problem need not be discussed at all, since as already pointed out, § 221, the stability mathematics referred to the wind is exactly the same as in still air, and, therefore, there is nothing new involved. An aeroplane which is stable in still air for given steady motion, is stable in a steady wind for the same relative steady motion.

226. **Longitudinal Motion.**—The really important problem of the wind is to find out *how* the aeroplane tends towards the new steady motion, especially if the velocity of the wind is at all considerable. We shall consider first the longitudinal motion of the symmetrical aeroplane with the wind in the plane of symmetry.

Let the wind have velocity components U_1', U_2' along the axes x, y. The opposite direction will be indicated by negative values of U_1', U_2' respectively. The air forces are defined by the motion of the aeroplane relatively to the wind, *i.e.* by the velocity components

$$U_1 - U_1' \cos \Theta - U_2' \sin \Theta,\ \ U_2 + U_1' \sin \Theta - U_2' \cos \Theta,$$

and the angular velocity $d\Theta/dt$. The equations of motion become

$$\frac{dU_1}{dt} - U_2 \frac{d\Theta}{dt} = g \sin \Theta + T - R_1,$$

$$\frac{dU_2}{dt} + U_1 \frac{d\Theta}{dt} = g \cos \Theta - R_2,$$

$$\frac{d^2\Theta}{dt^2} = - G_3,$$

where R_1, R_2, G_3, T are now the same functions of

$$U_1 - U_1' \cos \Theta - U_2' \sin \Theta,\ \ U_2 + U_1' \sin \Theta - U_2' \cos \Theta,\ \ d\Theta/dt,$$

as they are of U_1, U_2, $d\Theta/dt$ in the problem in still air. It need hardly be said that we cannot solve these equations in their general form, the chief difficulty being, as in the case of still air, the complicated and, at present, comparatively unknown forms of R_1, R_2, G_3, T.

There is one case in which some knowledge of the motion can be obtained, and this is when the wind is only a light breeze, so that U_1', U_2' are small compared to the speed of the aeroplane itself. If this is so we can expand, *e.g.*, R_1 in the form

$$R_1 = R_1(U_1,\ U_2,\ d\Theta/dt) + U\{(-U_1' \cos \Theta - U_2' \sin \Theta)a_x + (U_1' \sin \Theta - U_2' \cos \Theta)b_x\},$$

where $U = (U_1^2 + U_2^2)^{\frac{1}{2}}$ and a_x, b_x have meanings similar to those used in the work on stability. This involves the neglecting of squares and products of the ratios

$$(-U_1' \cos \Theta - U_2' \sin \Theta)/U,\ \ (U_1' \sin \Theta - U_2' \cos \Theta)/U.$$

Similar expressions can be used for R_2, G_3 and T. But as the effect of the wind is to cause changes in U_1, U_2, Θ, $d\Theta/dt$ also, we shall on the assumption of steady flight U_{10}, U_{20}, Θ_0 when the wind rises, write to the first order of small quantities

$$R_1 = R_{10} + U\{(u_1 - U_1' \cos \Theta_0 - U_2' \sin \Theta_0)a_x + (u_2 + U_1' \sin \Theta_0 - U_2' \cos \Theta_0)b_x + \omega_3 f_x\},$$

where

$$U_1 = U_{10} + u_1,\ \ U_2 = U_{20} + u_2,\ \ \Theta = \Theta_0 + \theta,\ \ \omega_3 = d\theta/dt,\ \ U = (U_{10}^2 + U_{20}^2)^{\frac{1}{2}},$$

and u_1, u_2, θ are small quantities of the same order as U_1', U_2'. Similar expressions can be used for R_2, G_3, whilst the T derivatives are supposed included in those for R_1, R_2, G_3. Hence to the first order of small quantities defined by the ratio of the velocity of the wind to that of the

aeroplane, we have the approximate equations of motion (in which the conditions of steady motion have been used):—

$$\left.\begin{aligned}
\frac{du_1}{dt} - U_{20}\frac{d\theta}{dt} &= g\theta\cos\Theta_0 - U\{(u_1 - U_1'\cos\Theta_0 - U_2'\sin\Theta_0)a_x \\
&\qquad + (u_2 + U_1'\sin\Theta_0 - U_2'\cos\Theta_0)b_x + \omega_3 f_x\}, \\
\frac{du_2}{dt} + U_{10}\frac{d\theta}{dt} &= -g\theta\sin\Theta_0 - U\{(u_1 - U_1'\cos\Theta_0 - U_2'\sin\Theta_0)a_y \\
&\qquad + (u_2 + U_1'\sin\Theta_0 - U_2'\cos\Theta_0)b_y + \omega_3 f_y\}, \\
\frac{d^2\theta}{dt^2} &= - U\{(u_1 - U_1'\cos\Theta_0 - U_2'\sin\Theta_0)a_3 \\
&\qquad + (u_2 + U_1'\sin\Theta_0 - U_2'\cos\Theta_0)b_3 + \omega_3 f_3\}.
\end{aligned}\right\} \quad (91)$$

Using the notation $D \equiv d/dt$, we get the equations

$$\left.\begin{aligned}
&(D + Ua_x)u_1 + (\quad Ub_x)u_2 + (\overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0)\theta \\
&\qquad = (a_x\cos\Theta_0 - b_x\sin\Theta_0)UU_1' + (a_x\sin\Theta_0 + b_x\cos\Theta_0)UU_2', \\
&(\quad Ua_y)u_1 + (D + Ub_y)u_2 + (\overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0)\theta \\
&\qquad = (a_y\cos\Theta_0 - b_y\sin\Theta_0)UU_1' + (a_y\sin\Theta_0 + b_y\cos\Theta_0)UU_2', \\
&(\quad Ua_3)u_1 + (\quad Ub_3)u_2 + (D^2 + Uf_3\,.\,D \qquad)\theta \\
&\qquad = (a_3\cos\Theta_0 - b_3\sin\Theta_0)UU_1' + (a_3\sin\Theta_0 + b_3\cos\Theta_0)UU_2'.
\end{aligned}\right\} \quad (92)$$

The expressions on the left-hand sides of the equations (92) are exactly the same as those given by the equations (104), Chapter III., whilst on the right-hand sides we have mere numbers, since U_1', U_2' are given constants.

227. Now if we suppose the quantities u_1, u_2, θ solved, we get, as in the theory of linear differential equations with constant coefficients, that each one consists of a part depending on the quantities on the right hand sides, the so-called *particular integral*, and a part depending on the operators on the left-hand sides, the *complementary function*. The latter must consist of multiples of $e^{U\lambda_1 t}$, $e^{U\lambda_2 t}$, $e^{U\lambda_3 t}$, $e^{U\lambda_4 t}$, where λ_1, λ_2, λ_3, λ_4 are the solutions of the algebraic equation (106), Chapter III. Hence in a *stable aeroplane* the complementary function will tend to disappear in each case, since all the real parts of the solutions, λ, are negative, and we shall be left with the particular integral alone. Thus, after a certain time, depending on how quickly the oscillations in the aeroplane are damped (Chapter VII., §§ 200–1), we shall have constant values of u_1, u_2, θ which define the new steady motion.

228. **Simplified Notation; Solution.**—At this stage a simplification suggests itself without any loss of generality. We have taken the wind velocity components to be U_1', U_2' horizontally and vertically. Let us take the wind velocity components along the X, Y directions in the body during the steady motion. These are definite fixed directions in space, and they have the advantage that the relative velocities of the aeroplane are simple expressions. Using the same symbols U_1', U_2' for these new components of the wind velocity, the reader will have no difficulty in satisfying himself that the equations (92) now become:—

$$\left.\begin{aligned}
&(D + Ua_x)u_1 + (\quad Ub_x)u_2 + (\overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0)\theta = a_x UU_1' + b_x UU_2', \\
&(\quad Ua_y)u_1 + (D + Ub_y)u_2 + (\overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0)\theta = a_y UU_1' + b_y UU_2', \\
&(\quad Ua_3)u_1 + (\quad Ub_3)u_2 + (D^2 + Uf_3\,.\,D \qquad)\theta = a_3 UU_1' + b_3 UU_2'.
\end{aligned}\right\} \quad (93)$$

In the case of a steady wind U_1', U_2' are constants, and we get

$$\left.\begin{array}{lllr}(D+Ua_x)(u_1-U_1')+(& Ub_x)(u_2-U_2')+(Uf_x-U_{20}\,.\,D-g\cos\Theta_0)\theta=0, \\ (\quad Ua_y)(u_1-U_1')+(D+ & Ub_y)(u_2-U_2')+(Uf_y+U_{10}\,.\,D+g\sin\Theta_0)\theta=0, \\ (\quad Ua_3)(u_1-U_1')+(& Ub_3)(u_2-U_2')+(D^2+Uf_3\,.\,D\qquad)\theta=0,\end{array}\right\} \quad (94)$$

verifying that the new steady motion is given by $u_1=U_1'$, $u_2=U_2'$: this means that the new steady motion of the aeroplane is simply the old steady motion together with the additional steady motion of the wind.

But now we also see that the general motion is given by

$$\left.\begin{array}{l}u_1=U_1'+A_1e^{U\lambda_1t}+A_2e^{U\lambda_2t}+A_3e^{U\lambda_3t}+A_4e^{U\lambda_4t}, \\ u_2=U_2'+B_1e^{U\lambda_1t}+B_2e^{U\lambda_2t}+B_3e^{U\lambda_3t}+B_4e^{U\lambda_4t}, \\ \theta\ = \qquad C_1e^{U\lambda_1t}+C_2e^{U\lambda_2t}+C_3e^{U\lambda_3t}+C_4e^{U\lambda_4t},\end{array}\right\} \quad . \quad . \quad . \quad (95)$$

where λ_1, λ_2, λ_3, λ_4 are the solutions of the algebraic equation (106) and A_1, A_2, . . . are arbitrary constants defined by the initial conditions. There are four initial conditions, namely

$$u_1=0,\ \ u_2=0,\ \ \theta=0,\ \ d\theta/dt=0\ ;$$

and, as in § 85, there are only four independent arbitrary constants in (95). The motion can thus be completely determined, assuming, of course, that the aeroplane is stable for the original steady motion. The aeroplane performs oscillations and then gradually assumes the new steady motion. It is merely a matter of arithmetic to find the values of the constants A_1, A_2, . . .

The motion of the aeroplane in space is easy to determine. Since θ is small, we can look upon u_1, u_2 as being referred to the original directions of the X, Y axes considered fixed in space.

The weakness of this method of small oscillations consists in the fact that it only applies to the case of a small wind velocity, and in actual flying practice wind velocities have to be negotiated which are considerable fractions of the velocity of flight. The method has, however, the great advantage that it can be applied easily to a *variable wind*.

229. **Variable Wind.**—It is immediately obvious that the analysis of § 226 applies to the case where U_1', U_2' are functions of the time, *so long as we assume that at any moment the air forces are such as would be given in steady motion of the same relative character.* As this assumption has been made all through our work there is nothing to be said against it here. Taking at once the simplified notation where U_1', U_2' are the component wind velocities (functions of the time) along the X, Y directions in the original steady motion, we have once more for small values of U_1', U_2' the approximate equations of motion (93), in which U_1', U_2' are small, and given functions of the time. We again have each quantity u_1, u_2, θ containing a particular integral and a complementary function, of which the latter is given by constant multiples of the time exponentials determined by the equation (106). The particular integrals are, however, more complicated now. There is no definite new steady motion to which the aeroplane tends. Its motion is determined by the nature of the wind fluctuations. At any moment the aeroplane tends to the steady motion appropriate to the instantaneous wind velocity; but at the next instant the appropriate steady motion is different, and the aeroplane tends to this fresh appropriate steady motion, carrying with it the effects of its past

tendencies. The motion is thus a continued attempt at steady motion, burdened by the inheritance of the whole of the past and the wind fluctuations of the immediate present.

To express the result mathematically, we must find the particular integrals, and for this purpose it is necessary to disentangle u_1, u_2, θ from the three equations (93). The process is somewhat long. It depends on the fact that *with constant coefficients* the operator $D(\equiv d/dt)$ can be treated as an algebraic symbol as far as multiplication is concerned. We can, therefore, deal with determinants containing D by expanding and then performing the operations indicated. It follows, *e.g.*, that

$$\begin{vmatrix} D+Ua_x, & Ub_x \\ Ua_y, & D+Ub_y \end{vmatrix} \equiv (D+Ua_x)(D+Ub_y) - Ub_x \cdot Ua_y$$
$$\equiv D^2 + U(a_x+b_y)D + U^2(a_xb_y - a_yb_x),$$

where D^2 means d^2/dt^2. The same applies to more complicated determinants. It follows that determinants containing the operator D can be manipulated in exactly the same way as ordinary algebraic determinants, and, in particular, that such a determinant, with two columns (or two rows) identical, vanishes completely, and that the quantity on which it operates disappears altogether.

Using this fact, let us operate on the equations (93) as follows:— operate on the first with the determinantal operator

$$\begin{vmatrix} D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ Ub_3, & D^2+Uf_3\cdot D \end{vmatrix};$$

on the second with

$$-\cdot\begin{vmatrix} Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ Ub_3, & D^2+Uf_3\cdot D \end{vmatrix};$$

on the third with

$$\begin{vmatrix} Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \end{vmatrix};$$

and add the resulting equations. We get the equation

$$\begin{vmatrix} D+Ua_x, & Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ Ua_y, & D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ Ua_3, & Ub_3, & D^2+Uf_3\cdot D \end{vmatrix} u_1$$

$$+\begin{vmatrix} Ub_x, & Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ D+Ub_y, & D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ Ub_3, & Ub_3, & D^2+Uf_3\cdot D \end{vmatrix} u_2$$

$$+\begin{vmatrix} \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0, & Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0, & D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ D^2+Uf_3\cdot D, & Ub_3, & D^2+Uf_3\cdot D \end{vmatrix} \theta$$

$$=\begin{vmatrix} Ua_x, & Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ Ua_y, & D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ Ua_3, & Ub_3, & D^2+Uf_3\cdot D \end{vmatrix} U_1'$$

$$+\begin{vmatrix} Ub_x, & Ub_x, & \overline{Uf_x-U_{20}}\cdot D - g\cos\Theta_0 \\ Ub_y, & D+Ub_y, & \overline{Uf_y+U_{10}}\cdot D + g\sin\Theta_0 \\ Ub_3, & Ub_3, & D^2+Uf_3\cdot D \end{vmatrix} U_2' \quad . \quad . \quad . \quad (96)$$

The quantities u_2, θ disappear altogether. The operator on U_2' can be simplified by subtracting the first column from the second. We get for u_1 the equation

$$\begin{vmatrix} D+Ua_x, & Ub_x, & \overline{Uf_x - U_{20}}\; D - g\cos\Theta_0 \\ Ua_y, & D+Ub_y, & \overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0 \\ Ua_3, & Ub_3, & D^2 + Uf_3\,.\,D \end{vmatrix} u_1$$

$$= \begin{vmatrix} Ua_x, & Ub_x, & \overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0 \\ Ua_y, & D+Ub_y, & \overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0 \\ Ua_3, & Ub_3, & D^2 + Uf_3\,.\,D \end{vmatrix} U_1'$$

$$+ D \begin{vmatrix} Ub_x, & \overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0 \\ Ub_3, & D^2 + Uf_3\,.\,D \end{vmatrix} U_2' \quad . \quad . \quad . \quad . \quad (97)$$

The operator acting on u_1 is

$$A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4,$$

where A_1, B_1, C_1, D_1, E_1 are the coefficients in the biquadratic for λ (106), Chapter III. We introduce an obvious notation for the operators on U_1', U_2' by writing them in the forms :—

$$(B_{x1}UD^3 + C_{x1}U^2D^2 + D_{x1}U^3D + E_{x1}U^4)\,U_1',$$
$$(B_{x2}UD^3 + C_{x2}U^2D^2 + D_{x2}U^3D \qquad\qquad)\,U_2',$$

where B_{x1}, B_{x2}, C_{x1}, C_{x2}, etc. are constants whose values are easily found from (97). The symbolism $B_{x1}U$, *e.g.*, means the coefficient of D^3 in operator on U_1' in the value of u_1; $C_{x1}U^2$ means the coefficient of D^2 in the operator on U_1' in the value of u_1; $B_{x2}U$ means the coefficient of D^3 in the operator on U_2' in the value of u_1; etc. We have then for u_1 the equation

$$\begin{aligned}(A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4)u_1 \\ = (B_{x1}UD^3 + C_{x1}U^2D^2 + D_{x1}U^3D + E_{x1}U^4)\,U_1' \\ + (B_{x2}UD^3 + C_{x2}U^2D^2 + D_{x2}U^3D)\,U_2' \quad . \quad . \quad (98)\end{aligned}$$

It is readily seen that $E_{x1} = E_1$.

In the same way we get

$$\begin{aligned}(A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4)u_2 \\ = (B_{y1}UD^3 + C_{y1}U^2D^2 + D_{y1}U^3D)\,U_1' \\ + (B_{y2}UD^3 + C_{y2}U^2D^2 + D_{y2}U^3D + E_{y2}U^4)\,U_2' \quad . \quad . \quad (99)\end{aligned}$$

where B_{y1}, C_{y1}, D_{y1}, B_{y2}, C_{y2}, D_{y2}, E_{y2} are now the coefficients in the operators

$$\begin{vmatrix} D+Ua_x, & Ua_x, & \overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0 \\ Ua_y, & Ua_y, & \overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0 \\ Ua_3, & Ua_3, & D^2 + Uf_3\,.\,D \end{vmatrix},$$

and

$$\begin{vmatrix} D+Ua_x, & Ub_x, & \overline{Uf_x - U_{20}}\,.\,D - g\cos\Theta_0 \\ Ua_y, & Ub_y, & \overline{Uf_y + U_{10}}\,.\,D + g\sin\Theta_0 \\ Ua_3, & Ub_3, & D^2 + Uf_3\,.\,D \end{vmatrix}$$

respectively; the former can of course be simplified by subtracting the second column from the first. We have $E_{y2} = E_1$.

Finally, we have

$$(A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4)\theta$$
$$= (C_{31}U^2D^2 + D_{31}U^3D)U_1' + (C_{32}U^2D^2 + D_{32}U^3D)U_3' \quad . \quad . \quad (100)$$

where C_{31}, D_{31}, C_{32}, D_{32} are the coefficients in the operators

$$\begin{vmatrix} D+Ua_x, & Ub_x, & Ua_x \\ Ua_y, & D+Ub_y, & Ua_y \\ Ua_3, & Ub_3, & Ua_3 \end{vmatrix}, \qquad \begin{vmatrix} D+Ua_x, & Ub_x, & Ub_x \\ Ua_y, & D+Ub_y, & Ub_y \\ Ua_3, & Ub_3, & Ub_3 \end{vmatrix}$$

respectively. These operators easily reduce to

$$D\begin{vmatrix} D+Ub_y, & Ua_y \\ Ub_3, & Ua_3 \end{vmatrix}, \qquad D\begin{vmatrix} D+Ua_x, & Ub_x \\ Ua_3, & Ub_3 \end{vmatrix}.$$

It will be noticed that the equation for θ has only second powers of D on the right-hand side, whereas the equations for u_1, u_2 have third powers of D on their right-hand sides.

We have thus disentangled u_1, u_2, θ, and each is given by means of a linear differential equation with constant coefficients, in which the right-hand sides are known functions of the time, since presumably U_1', U_2' are known functions of the time. It follows that each consists of a complementary function in terms of the time exponentials so often used in this work, and of a particular integral. We can, therefore, study the motion of a stable aeroplane due to any given moderate wind fluctuations.

230. **Forced Motions.**—In any dynamical problem the most complicated arithmetic usually occurs in the finding of the values of the arbitrary constants from the given initial conditions. In the case of a sudden change of wind from one form of steady blow to another form, there is nothing else of interest in connection with the problem of how the aeroplane moves because of the change. In the problem of a fluctuating wind we shall omit all reference to the initial conditions, because the particular integrals contain so much that is interesting. We can consider the particular integrals as defining the *forced* motion of the aeroplane, when all its natural inherent oscillations have been practically damped out because of its stability.

231. **Wind Rising Gradually.**—A useful problem to investigate is the case where the wind rises gradually, starting from zero and approaching a constant value asymptotically. To form a definite picture of such a case we take (as suggested by E. B. Wilson)

$$\left.\begin{aligned} U_1' &= J_1(1 - e^{-Uat}) \\ U_2' &= J_2(1 - e^{-Uat}) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

a being a positive real number, so that we have a wind constant in direction, and ultimately of velocity $(J_1^2 + J_2^2)^{\frac{1}{2}}$. The velocity in terms of the time is indicated graphically in Fig. 119.

Equation (98) now gives

$$(A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4)u_1$$
$$= J_1E_{x1}U^4 + J_1(B_{x1}a^3 - C_{x1}a^2 + D_{x1}a - E_{x1})U^4e^{-Uat}$$
$$+ J_2(B_{x2}a^3 - C_{x2}a^2 + D_{x2}a \qquad)U^4e^{-Uat},$$

so that after the oscillations have been damped out, we have

$$\left.\begin{aligned} u_1 &= J_1 + \frac{J_1(B_{x1}a^3 - C_{x1}a^2 + D_{x1}a - E_{x1}) + J_2(B_{x2}a^3 - C_{x2}a^2 + D_{x2}a)}{A_1a^4 - B_1a^3 + C_1a^2 - D_1a + E_1}\,e^{-Uat}, \\ &\text{Similarly} \\ u_2 &= J_2 + \frac{J_1(B_{y1}a^3 - C_{y1}a^2 + D_{y1}a) + J_2(B_{y2}a^3 - C_{y2}a^2 + D_{y2}a - E_{y2})}{A_1a^4 - B_1a^3 + C_1a^2 - D_1a + E_1}\,e^{-Uat}, \\ &\text{and} \\ \theta &= \frac{J_1(-C_{31}a^2 + D_{31}a) + J_2(-C_{32}a^2 + D_{32}a)}{A_1a^4 - B_1a^3 + C_1a^2 - D_1a + E_1}\,e^{-Uat}, \end{aligned}\right\} \quad (102)$$

it being assumed that the exponential e^{-Uat} decreases very slowly compared to the exponentials $e^{U\lambda t}$ in the free oscillations of the aeroplane. We can

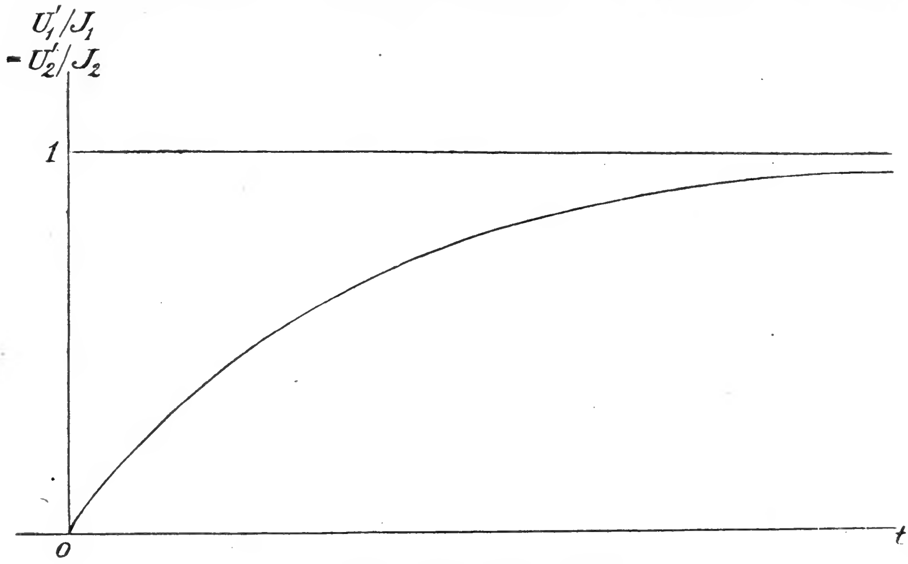

FIG. 119.—Gradually Rising Wind.

ultimately omit the term e^{-Uat} in each case, since this too becomes small. The final effect of the wind (101) is, therefore, to give

$$u_1 = J_1, \quad u_2 = J_2, \quad \theta = 0,$$

the modified steady motion, as is to be expected.

In view of what follows, we must remark that the results (102) are true, because in a stable aeroplane no real value of a can be found to make the denominator

$$A_1a^4 - B_1a^3 + C_1a^2 - D_1a + E_1$$

vanish.

232. **Periodic Wind.**—Another interesting kind of wind fluctuation is when the wind is in the form of a series of puffs of equal strengths. Since any periodic quantity can be expressed as the sum of sines and cosines by means of a Fourier expansion, we take as a typical case

$$U_1' = J_1 \sin U\beta t, \quad U_2' = J_2 \sin U\beta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where the wind has constant direction defined by the ratio J_1/J_2 and varies harmonically with period $2\pi/\beta U$ seconds. We suppose the time measured from an instant of no wind ($U_1' = U_2' = 0$).

If we refer to the general forms of the equations for u_1, u_2, θ, we see that it is reasonable to suggest for the particular integrals, *i.e.* for the *forced oscillations maintained by the wind fluctuations*, the following:—

$$\left.\begin{aligned} u_1 &= u_{11}\sin U\beta t + u_{12}\cos U\beta t, \\ u_2 &= u_{21}\sin U\beta t + u_{22}\cos U\beta t, \\ \theta &= \theta_1 \sin U\beta t + \theta_2 \cos U\beta t, \end{aligned}\right\} \qquad (104)$$

u_{11}, u_{12}, etc., being certain constants. We determine these constants by substituting in equations (98–100) and equating coefficients of $\sin U\beta t$, $\cos U\beta t$ on the two sides of each equation, since these equations are true for all values of t. Thus equation (98) gives

$$\begin{aligned} &\{(A_1\beta^4 - C_1\beta^2 + E_1)u_{11} + (B_1\beta^3 - D_1\beta)u_{12}\}\sin U\beta t \\ &\qquad + \{(A_1\beta^4 - C_1\beta^2 + E_1)u_{12} - (B_1\beta^3 - D_1\beta)u_{11}\}\cos U\beta t \\ &= \{(-C_{x1}\beta^2 + E_{x1})J_1 + (-C_{x2}\beta^2)J_2\}\sin U\beta t \\ &\qquad + \{(-B_{x1}\beta^3 + D_{x1}\beta)J_1 + (-B_{x2}\beta^3 + D_{x2}\beta)J_2\}\cos U\beta t. \end{aligned}$$

Hence, by equating coefficients, we get

$$\begin{aligned} &(A_1\beta^4 - C_1\beta^2 + E_1)u_{11} + (B_1\beta^3 - D_1\beta)u_{12} \\ &\qquad = (-C_{x1}\beta^2 + E_{x1})J_1 + (-C_{x2}\beta^2)J_2, \\ &(A_1\beta^4 - C_1\beta^2 + E_1)u_{12} - (B_1\beta^3 - D_1\beta)u_{11} \\ &\qquad = (-B_{x1}\beta^3 + D_{x1}\beta)J_1 + (-B_{x2}\beta^3 + D_{x2}\beta)J_2, \end{aligned}$$

whence we deduce

$$\left.\begin{aligned} u_{11} &= \frac{\begin{aligned}[\{&(A_1\beta^4 - C_1\beta^2 + E_1)(-C_{x1}\beta^2 + E_{x1}) - (B_1\beta^3 - D_1\beta)(-B_{x1}\beta^3 + D_{x1}\beta)\}J_1 \\ &+ \{(A_1\beta^4 - C_1\beta^2 + E_1)(-C_{x2}\beta^2) - (B_1\beta^3 - D_1\beta)(-B_{x2}\beta^3 + D_{x2}\beta)\}J_2]\end{aligned}}{(A_1\beta^4 - C_1\beta^2 + E_1)^2 + (B_1\beta^3 - D_1\beta)^2}, \\ u_{12} &= \frac{\begin{aligned}[\{&(B_1\beta^3 - D_1\beta)(-C_{x1}\beta^2 + E_{x1}) + (A_1\beta^4 - C_1\beta^2 + E_1)(-B_{x1}\beta^3 + D_{x1}\beta)\}J_1 \\ &+ \{(B_1\beta^3 - D_1\beta)(-C_{x2}\beta^2) + (A_1\beta^4 - C_1\beta^2 + E_1)(-B_{x2}\beta^3 + D_{x2}\beta)\}J_2]\end{aligned}}{(A_1\beta^4 - C_1\beta^2 + E_1)^2 + (B_1\beta^3 - D_1\beta)^2}. \end{aligned}\right\} \quad (105)$$

To obtain u_{21}, u_{22} from u_{11}, u_{12} we change the suffix x into suffix y; and to obtain θ_1, θ_2 from u_{11}, u_{12} we change the suffix x into suffix 3; in each case we, of course, drop those coefficients that happen to be zero.

As far as practical flight is concerned, the inconvenient effect of the fluctuating wind is to cause oscillations in the direction of the machine. This is indicated in the value of θ, for which we have

$$\left.\begin{aligned} \theta &= \frac{\begin{aligned}[\{&(A_1\beta^4 - C_1\beta^2 + E_1)(-C_{31}\beta^2) - (B_1\beta^3 - D_1\beta)(D_{31}\beta)\}J_1 \\ &+ \{(A_1\beta^4 - C_1\beta^2 + E_1)(-C_{32}\beta^2) - (B_1\beta^3 - D_1\beta)(D_{32}\beta)\}J_2]\end{aligned}}{(A_1\beta^4 - C_1\beta^2 + E_1)^2 + (B_1\beta^3 - D_1\beta)^2}\sin U\beta t \\ &+ \frac{\begin{aligned}[\{&(B_1\beta^3 - D_1\beta)(-C_{31}\beta^2) + (A_1\beta^4 - C_1\beta^2 + E_1)(D_{31}\beta)\}J_1 \\ &+ \{(B_1\beta^3 - D_1\beta)(-C_{32}\beta^2) + (A_1\beta^4 - C_1\beta^2 + E_1)(D_{32}\beta)\}J_2]\end{aligned}}{(A_1\beta^4 - C_1\beta^2 + E_1)^2 + (B_1\beta^3 - D_1\beta)^2}\cos U\beta t. \end{aligned}\right\} \quad (106)$$

233. **Desirable Qualities of an Aeroplane.**—It would be difficult to discuss the physical meaning of this result for the general value of β;

but we can discuss the requisites of a good aeroplane by considering two special cases :—

(i) β small, so that the period of the wind fluctuation is very long :

(ii) β large, so that the period is very short.

(i) If β is small, we omit squares and higher powers of β when occurring in conjunction with β itself, and we find from (106)

$$\theta = \frac{D_{31}J_1 + D_{32}J_2}{E_1}\beta \cos U\beta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

where, by (100),

$$D_{31} = (b_y a_3 - a_y b_3)/U,$$
$$D_{32} = (a_x b_3 - b_x a_3)/U.$$

We see by (107) that it is important to have D_{31}/E_1 and D_{32}/E_2 as small as possible.

For the idealised aeroplane, § 196, we have

$$a_3 = 0, \ E_1 = \frac{2g^2lKS_2}{CU^4}.$$

Hence

$$\frac{D_{31}}{E_1} = \frac{2glKS_2}{CU^3}\bigg/\frac{2g^2lKS_2}{CU^4} = \frac{U}{g},$$

$$\frac{D_{32}}{E_1} = -\frac{2glKS_2 \tan a}{CU^3}\bigg/\frac{2g^2lKS_2}{CU^4} = -\frac{U \tan a}{g}.$$

(The variation due to the propeller is omitted in this approximate calculation.) It follows that

$$\theta = (J_1 - J_2 \tan a)\frac{U\beta}{g}\cos U\beta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

Thus θ is always small. It is interesting to notice that the main effect is due to the X component of the wind.

(ii) If β is large, we use only the highest powers of β in (106), so that (as $A_1 = 1$)

$$\theta = -\frac{C_{31}J_1 + C_{32}J_2}{\beta^2}\sin U\beta t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

where, by (100),

$$C_{31} = a_3/U, \ C_{32} = b_3/U.$$

Using § 196, we see that in the idealised aeroplane there considered $C_{31} = 0$, whilst

$$C_{32} = -\frac{lKS_2}{CU}.$$

Hence

$$\theta = \frac{lKS_2J_2}{C\beta^2U}\sin U\beta t.$$

Thus for short period fluctuations in the wind, the main effect in θ is due to the Y component of the wind. We must have this effect as small as possible.

Now by § 196, equation (18), we have

$$mg = KS_1U^2 \sin a \cos a,$$

whilst

$$C = mk_3{}^2 = \frac{k_3{}^2}{g} KS_1U^2 \sin a \cos a.$$

Hence

$$\theta = \frac{glS_2}{k_3{}^2S_1 \sin a \cos a} \frac{J_2}{\beta^2U^3} \sin U\beta t \quad . \quad . \quad . \quad . \quad . \quad (110)$$

We see, then, that l and S_2 must not be too large. This means that the tail plane must not be too large and not too far behind the main plane of the machine.

It is also clear that the effect on θ is small if U is large. This could have been predicted by common sense.

234. **Margin of Stability should be Moderate.**—This result is of significance. We have seen that to get stability we need to have $lS_2/k_3{}^2S_1$ large enough to satisfy the condition (24), § 198. It would, therefore, seem reasonable to make the margin of stability a generous one. Our present result shows, however, that this is not the case. Too great a margin of stability makes the machine sensitive to wind fluctuations. This explains the objections raised by practical fliers to the stable machine in the early history of the development of flight. They found a stable machine more difficult to manipulate and to control in gusty weather. This is avoided by restricting oneself to a comparatively small margin of stability.

It is, nevertheless, important to take care to get a sufficient margin. For if Routh's discriminant is too near zero, then for a value of β equal to that for the short period free oscillation of the aeroplane, the denominator in θ can become very near zero, and then the effect of the wind can become very marked. If, as a limiting case, the discriminant is exactly zero, and β happens to make the denominator zero, we have to solve the equations (98–100) by a slightly different process, and we get a factor t in u_1, u_2 and θ, showing that the **resonance** causes continuously increasing amplitude in the free oscillations, and we get instability.

235. **Damped Periodic Wind.**—When the periodic wind is itself damped, so that we have a succession of gusts of decreasing strength, we can write

$$\left.\begin{aligned} U_1' &= J_1e^{-Uat} \sin U\beta t, \\ U_2' &= J_2e^{-Uat} \sin U\beta t. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

(Care must be taken to distinguish between the a here and the angle of attack in the idealised aeroplane.) We now use the expressions

$$\left.\begin{aligned} u_1 &= (u_{11} \sin U\beta t + u_{12} \cos U\beta t)e^{-Uat}, \\ u_2 &= (u_{21} \sin U\beta t + u_{22} \cos U\beta t)e^{-Uat}, \\ \theta &= (\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)e^{-Uat}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

and obtain the values of u_{11}, u_{12}, etc., by substituting in the equations (98–100) and equating the coefficients of $\sin \beta t$, $\cos \beta t$, on the two sides of each equation.

For θ_1, θ_2, we obtain

$$\begin{aligned}\{A_1(D-Ua)^4 + B_1U(D-Ua)^3 + C_1U^2(D-Ua)^2 + D_1U^3(D-Ua)&\\ + E_1U^4\}(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)&\\ = \{C_{31}U^2(D-Ua)^2 + D_{31}U^3(D-Ua)\}(J_1 \sin U\beta t)&\\ + \{C_{32}U^2(D-Ua)^2 + D_{32}U^3(D-Ua)\}(J_2 \sin U\beta t), \quad . \quad . \quad . \quad (113)&\end{aligned}$$

using the well known fact

$$f(D)\{e^{at}\phi(t)\} = e^{at}f(D+a)\{\phi(t)\},$$

where $f(D)$ is an algebraic expression in D, with constant coefficients, and a is mere number.

The student can, as an exercise, work out the linear equations for θ_1, θ_2, solve, and then deduce the value of θ. By considering the cases β small and β large respectively (examining, in the first case, the effects of a small and large, *i.e.* slow and quick damping), he will easily deduce once again that a fair margin of stability should exist, but not too large a margin.

236. **Damped Periodic Wind of the Same Complex Period as One of the Free Periods of the Aeroplane.**—When the periodic wind is damped, the interesting special case arises where the complex exponent representing the time variation of the gust agrees with one of the complex exponents representing the free damped oscillations of the stable aeroplane. This means that $-a \pm \iota\beta$ are solutions of the algebraic equation

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0.$$

It is unnecessary to work out this case in full: the algebra would be very intricate. We can discover all we want to know by using the fact that in such a case the terms

$$(u_{11} \sin U\beta t + u_{12} \cos U\beta t)e^{-Uat}, \text{ etc.}$$

are really parts of the complementary functions, and are actually of little interest in this case. To obtain the particular integrals, we now use

$$\left.\begin{aligned} u_1 &= (u_{11} \sin U\beta t + u_{12} \cos U\beta t)te^{-Uat},\\ u_2 &= (u_{21} \sin U\beta t + u_{22} \cos U\beta t)te^{-Uat},\\ \theta &= (\theta_1 \ \sin U\beta t + \theta_2 \ \cos U\beta t)te^{-Uat};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

and the coefficients u_{11}, u_{12}, etc., can again be found by substitution in (98–100).

We have, then, forced oscillations with amplitudes varying as the quantity

$$a \equiv te^{-Uat}.$$

Now

$$\frac{da}{dt} = (1 - Uat)e^{-Uat},$$

and

$$\frac{d^2a}{dt^2} = Ua(Uat - 2)e^{-Uat}.$$

Therefore, the amplitude is a *maximum* at $t = 1/Ua$, since this makes $da/dt = 0$, d^2a/dt^2, negative. Using $t = 1/Ua$, we find that the maximum value of a is $1/Uae$. Hence a possible danger arises from small values of

a, *i.e.* for slow damping. In a stable machine there is a slowly damped oscillation (Chapter VII., § 201). Thus there is a danger of large θ when approximately

$$Ua = \frac{C_1D_1 - E_1B_1}{2C_1{}^2}, \quad U\beta = \left(\frac{E_1}{C_1}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

In the machine considered in § 199, defined by equation (27), if

$$Ua = \tfrac{1}{22}, \quad U\beta = \tfrac{2}{7} \text{ nearly},$$

then the amplitude rises to a maximum of eight times the initial value.

It is, therefore, necessary to investigate the case when the complex period of the wind fluctuation is the same as that of the slow free oscillation of the aeroplane. Putting in (100) the value of θ in (114), we get

$$\begin{aligned}\{A_1(D-Ua)^4 + B_1U(D-Ua)^3 + C_1U^2(D-Ua)^2 + D_1U^3(D-Ua)& \\ + E_1U^4\}\{t(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)\}& \\ = \{C_{31}U^2(D-Ua)^2 + D_{31}U^3(D-Ua)\}(J_1 \sin U\beta t)& \\ + \{C_{32}U^2(D-Ua)^2 + D_{32}U^3(D-Ua)\}(J_2 \sin U\beta t).& \quad . \quad . \quad . \quad (116)\end{aligned}$$

Let us write

$$\begin{aligned}A_1(D-Ua)^4 + B_1U(D-Ua)^3 + C_1U^2(D-Ua)^2 + D_1U^3(D-Ua) + E_1U^4& \\ \equiv A_1D^4 + B_1'UD^3 + C_1'U^2D^2 + D_1'U^3D + E_1'U^4;&\end{aligned}$$

then

$$\begin{aligned}B_1' = B_1 - 4A_1a, \quad C_1' = C_1 - 3B_1a + 6A_1a^2,& \\ D_1' = D_1 - 2C_1a + 3B_1a^2 - 4A_1a^3.& \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)\end{aligned}$$

Also

$$\begin{aligned}&D^n\{t(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)\} \\ &= tD^n(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t) + nD^{n-1}(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)\end{aligned}$$

by Leibnitz' Theorem. Further a, β are such that

$$(A_1D^4 + B_1UD^3 + C_1U^2D^2 + D_1U^3D + E_1U^4)\{(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t)e^{-Uat}\} = 0.$$

Hence we get from (116)

$$\begin{aligned}&(4A_1D^3 + 3B_1'UD^2 + 2C_1'U^2D + D_1'U^3)(\theta_1 \sin U\beta t + \theta_2 \cos U\beta t) \\ &= \{C_{31}U^2D^2 + (D_{31} - 2C_{31}a)U^3D + (-D_{31}a + C_{31}a^2)U^4\}(J_1 \sin U\beta t) \\ &- \{C_{32}U^2D^2 + (D_{32} - 2C_{32}a)U^3D + (-D_{32}a + C_{32}a^2)U^4\}(J_2 \sin U\beta t). \quad . \quad . \quad (118)\end{aligned}$$

If θ_1, θ_2 are found and account is taken of the fact that (Ua) and $(U\beta)^2$ are small quantities of the same order, we find that in order to avoid a large disturbance in this case, the machine being the idealised aeroplane of § 196, the margin of stability must be considerable, and the tail plane must be fairly far behind the main plane.

237. **General Theory.**—It is clear that the method of this chapter can be immediately applied to any wind in any direction, and for any machine, symmetrical or non-symmetrical. The student will have no difficulty in investigating further problems, particularly with regard to lateral gusts for a symmetrical aeroplane, a cyclonic wind in the longitudinal plane, etc. In fact, the problem of the behaviour of an aeroplane

in a wind, or under any disturbance, is one that offers much scope for further advance. The following references will be found useful:—

E. B. Wilson, "Theory of an Aeroplane Encountering Gusts," *Proc. Nat. Acad. of Sciences*, U.S.A., May, 1916, pp. 294–7.

L. Bairstow and J. L. Nayler, *Technical Report of the Advisory Committee for Aeronautics*, 1913–14, pp. 235–270.

S. Brodetsky, *Aeronautical Journal*, 1916.

In the last paper will be found an investigation based on the method of initial accelerations, as well as further details on the method of small oscillations.

EXERCISES

1. A sudden gust of wind strikes an aeroplane longitudinally. Find the change in motion produced (see Exercise 9, Chapter VII.).
2. Examine the motion of an aeroplane with a longitudinal cyclonic wind of constant strength: take $U_1' = U' \cos \omega t$, $U_2' = U' \sin \omega t$.
3. Investigate the case of a cyclonic wind whose strength diminishes exponentially.
4. An aeroplane descends with given motion relatively to the air. Show that in a steady wind the points where it reaches the ground are approximately on a circle.
5. Calculate the direction in which an aeroplane must be pointed so that it should fly in a given direction, in a given wind.
6. Examine the path of an aeroplane in circling flight, in a steady horizontal wind. How is the rotation affected?
7. Investigate a spiral glide in a steady horizontal wind.

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